Efficient Realization of Nonzero Spetra by Polynomial Matrices

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Outline

The Nonnegative Inverse Eigenvalue Problem

Definition

The **spectrum** of a matrix A, $sp(A) = (\lambda_1, \lambda_2, \cdots \lambda_n)$ is the set (with multiplicity) of the eigenvalues of the matrix A.

The **characteristic polynomial** of A is:

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When such a matrix A exists we say A **realizes** σ, and σ is **realizable**.

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There are several known necessary conditions for σ to be realizable by a primitive matrix:

1. $\exists \lambda_i \in \sigma$ such that $\lambda_i \in \mathbb{R}_+$ and $\lambda_i > |\lambda_j| \;\; j \neq i$. (Due to the Perron-Frobenius Theorem. We refer to λ_i as the **Perron** eigenvalue or root.)

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2. $\sigma = \bar{\sigma}$ (For each complex number in σ, its complex conjugate is also in σ.)
3. The kth moment of σ, $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$. $\forall k \in \mathbb{N}$ and if $s_k > 0$ then $s_{nk} > 0 \ \ \forall n \in \mathbb{N}$ (since s_k would be the trace of the matrix A^k)

Example (n = 2)

Let $n = 2$, $\sigma = (\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 > |\lambda_2|$.
Then σ is realized by the matrix:

$$A = \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix}$$

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Necessary and Sufficient conditions are known only when $n \leq 3$.

The Boyle Handelman Theorem

Theorem (Boyle and Handelman, 1991)

Let σ satisfy the previous necessary conditions. Then $\exists N \in \mathbb{N}$ such that σ augmented by N zeros (ie $\sigma' = (\lambda_1, \lambda_2, \cdots \lambda_n, 0, \cdots 0)$) is realizable by a primitive matrix.

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Given an n-tuple $\sigma := (\lambda_1, \lambda_2, \cdots \lambda_n)$ $\lambda_i \in \mathbb{C} \backslash \{0\}$ The Boyle Handleman theorem gives the necessary conditions for σ to be the **nonzero spectrum** of some matrix A, but the proof is not constructive, and puts no bounds on the size of this matrix.

The BH theorem and Characteristic Polynomials

Given a polynomial $p(t)$ the Boyle Handelman theorem specifies when there exists a primitive matrix A and natural number N such that:

$$t^N p(t) = \chi_A(t) = det(It - A) = \prod_{i=1}^{n}(t - \lambda_i)$$

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Alternatively, one can look at:

$$\chi_A^{-1}(t) = det(I - tA) = \prod_{i=1}^{n} (1 - t\lambda_i).$$

This **reverse characteristic polynomial** does not change as additional zero eigenvalues are added. Thus the Boyle Handelman theorem specifies when a given polynomial is exactly the reverse characteristic polynomial of some matrix A, but puts no bound on the size of A.

Graphs and Polynomial Matrices

Any matrix A over $\mathbb{R}_+$ can be treated as the adjacency matrix for some directed graph G in which the entry in position (i, j) is the weight of the edge from vertex i to vertex j.

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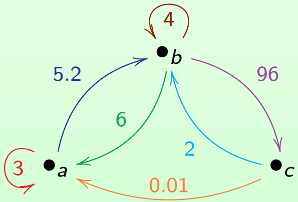
$$\begin{bmatrix} 3 & 5.2 & 0 \\ 6 & 4 & 96 \\ 0.01 & 2 & 0 \end{bmatrix}$$



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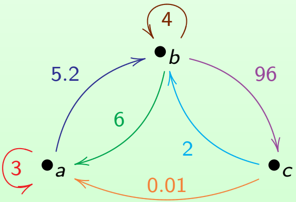

G can also be represented by a polynomial matrix $M(t)$ over $t\mathbb{R}_+[t]$.

Graphs and Polynomial Matrices

Construction of G from $M(t)$:
Let $M(t)$ be an $N \times N$ matrix over $t\mathbb{R}_+[t]$.

1. Assign N vertices the labels 1,2,...N.
2. For each term wt^p of the polynomial in the (i,j) position of A[t], construct a path of length p from vertex i to j with p-1 new distinct vertices.
3. Weight the first edge w and each additional edge 1(if $p > 1$.)

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In constructing a polynomial matrix from a graph, the weights of consecutive edges through "unimportant" vertices are multiplied to find the term's coefficient.

Graphs and Polynomial Matrices

Example:

$$\begin{bmatrix} 5t^3 + 1.5t & 9t^3 & 0 \\ \pi t^2 & 0 & 4t^2 \\ 2t & 0.3t^2 + t & 3.6t \end{bmatrix}$$

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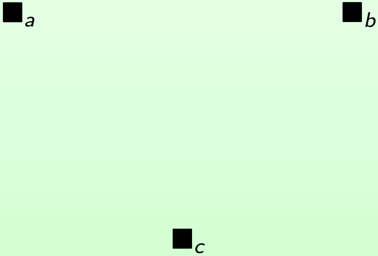

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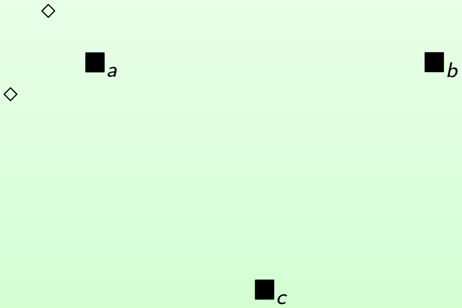

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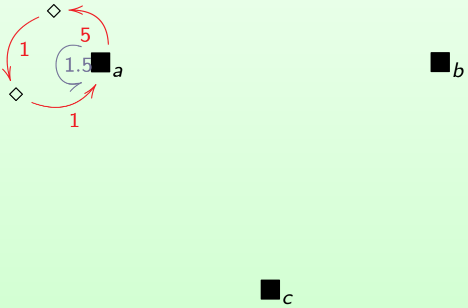

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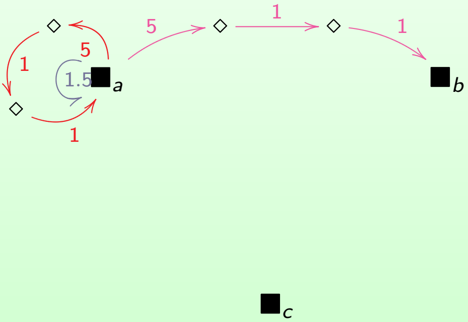

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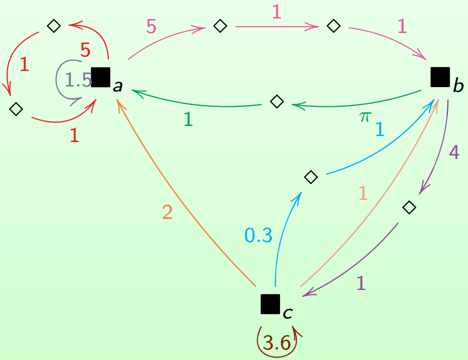

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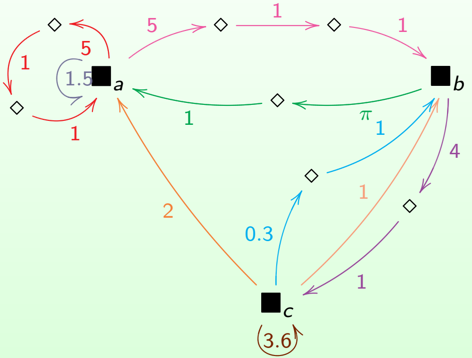

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1.5 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3.6 & 0 & 0.3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi & 0 & 0 & 4 & 0 \end{bmatrix}$$

$$\chi_G(t) = t^{10} - 5.1t^9 + 5.4t^8 - 9t^7 + 22.8t^6 + (1.8 - 9\pi)t^5 + (32.4\pi - 52)t^4 + 6t^3$$

$$\chi_G^{-1}(t) = 6t^7 + (32.4\pi - 52)t^6 + (1.8 - 9\pi)t^5 + 22.8t^4 - 9t^3 + 5.4t^2 - 5.1t + 1$$

Graphs and Polynomial Matrices

Theorem

Given two Matricies A over $\mathbb{R}_+$ and $M(t)$ over $t\mathbb{R}_+[t]$ that correspond to the same graph G, then:

$$\chi_M^{-1}(t) = det(I - At) = det(I - M(t))$$

Proof.

Use row operations on $I - At$ to combine rows/columns along a path, followed by expansion by minors to transform $I - At$ into $I - M(t)$ without changing the determinant. □

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A graph with nonnegative entries can be used to describe the possible trajectories of a dynamical system (Symbolic Dynamics)

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In the case of a **Shift of Finite Type**, all of the information about the dynamical system is encoded in its zeta function, which corresponds to the characteristic polynomial of its graph.

When is a given polynomial the characteristic polynomial (zeta function) for some shift of finite type?

A Theorem

Theorem

Assume that $p(t) = \prod_{i=1}^{d}(1 - \lambda_i t)$ *where the* $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ *satisfy the conditions:*

1. $\exists \lambda_i \in \sigma$ *such that* $\lambda_i \in \mathbb{R}_+$ *and* $\lambda_i > |\lambda_j|\ \ j \neq i$.
2. $\sigma = \bar{\sigma}$
3. $s_k = \sum_{i=1}^{n} \lambda_i^k > 0.\ \ \forall k \in \mathbb{N}$

Then there is an $N \geq 1$ *such that the power series expansion for* $p(t)^{1/N}$ *is of the form*

$$p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k$$

where $r_k \geq 0$ *for all* $k \geq 1$.

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Given a polynomial $p(t)$ with $p(0) = 1$, when does there exist a polynomial matrix $A(t) \in t\mathbb{R}^+[t]$ such that

$$p(t) = det(I - A(t))$$

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Our goal: Reprove the Boyle Handelman theorem in a constructive way, putting some bound on the size of the polynomial matrix necessary to realize a polynomial.

The Conjecture

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Let $p(t)$ be a polynomial which satisfies the condition that $\exists N \geq 1$ such that $p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k$ where $r_k \geq 0$ for all $k \geq 1$.

Then there exists an $N \times N$ polynomial matrix $M[t]$ with all nonnegative coefficients such that $\det(I - M[t]) = p(t)$.

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Results so far: This conjecture is true for N=1,2,3.

Case N = 1

Proof (N=1).

Trivial. If $p(t)^1 = 1 - r(t)$ where r(t) has no negative coefficients then the matrix A(t) = [r(t)] suffices.

$$det(I - A(t)) = det([1 - r(t)]) = 1 - r(t) = p(t)$$

□

N=2

Proof (N=2).

Suppose $p(t)^{1/2} = 1 - r(t)$ where r(t) has no negative coefficients. Then let q(t) be the polynomial that results when r(t) is truncated to some degree n (greater than or equal to the degree of p(t).)

Consider the polynomial $(1 - q(t))^2$.
The first "incomplete" term has order n+1, so the first n coefficients match p(t). Let $R(t) = (1 - q(t))^2 - p(t)$. Then:

$$R(t) = \sum_{i=n+1}^{2n} \sum_{j+k=i} q_j q_k t^i$$

Since all q_j and q_k are nonnegative, R(t) will contain only nonnegative terms.

Proof Continued(N=2).

Then construct the matrix:

$$A(t) = \begin{bmatrix} q(t) & \frac{R(t)}{t} \\ t & q(t) \end{bmatrix}$$

$$det(I - A(t)) = (1 - q(t))^2 - R(t) = p(t)$$

□

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$$A(t) = \begin{bmatrix} \left(\frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}\right) & \left(\frac{517t^3}{64} + \frac{323t^4}{64} + \frac{361t^5}{256}\right) \\ (t) & \left(\frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}\right) \end{bmatrix}$$

Idea:
Again, suppose $p(t)^{1/3} = 1 - r(t)$ where r(t) has no negative coefficients, let q(t) be r(t) truncated to degree n, and let s(t) be the remainder, so r(t) = q(t) + s(t).

$$A(t) = \begin{bmatrix} q(t) & \alpha(t) & \beta(t) \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix}$$

$$det(I - A(t)) = (1 - q(t))^3 - t^2\alpha(t) + t\beta(t)(1 - q(t))$$

This time $R(t) = (1 - q(t))^3 - p(t)$ is not strictly positive.

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$$det(I - A(t)) = (1 - q(t))^3 - t^2\alpha(t) + t\beta(t)(1 - q(t))$$

$p(t) = 1 - 5t + 7t^2 - 3t^3$

$p(t)^{1/2} = 1 - \frac{5t}{2} + \frac{3t^2}{8} - \frac{9t^3}{16} - \frac{189t^4}{128} - \frac{891t^5}{256} \dots$

$p(t)^{1/3} = 1 - \frac{5t}{3} - \frac{4t^2}{9} - \frac{76t^3}{81} - \frac{508t^4}{243} - \frac{3548t^5}{729} \dots$

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$(1 - q(t))^3 - \frac{508t^4}{81}(1 - q(t))$

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...

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An algorithm:
By repeatedly taking the lowest "remainder" term, construct:

$$b(t) = \sum_{i=M+1}^{3n} b_i t^i$$

such that $p(t) - (1 - q(t))^3 - b(t)(1 - q(t))$ has coefficient 0 for all terms with degree $3n$ or less.

N=3

We can calculate the coefficients of b:

$$b_m = 3\left[s(t)(1-q(t)-s(t))\right]_m = 3\left[r_m + \sum_{i=1}^{m-n} r_i r_m - i\right]$$

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There exists M such that if we truncate $r(t)$ to order $n \geq M$, the polynomial $b(t)$ has no negative coefficients.

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Proof.

Lots of careful approximations of binomial coefficients. □

Proof (N=3).

Choose n such that the proposition holds, let q(t) be the power series of p(t) to degree n, and construct b(t) as before.

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$$A(t) = \begin{bmatrix} q(t) & \alpha(t) & \beta(t) \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix}$$

$$\begin{aligned} det(I - A(t)) &= (1 - q(t))^3 - t^2\alpha(t) + t\beta(t)(1 - q(t)) \\ &= (1 - q(t))^3 - (p(t) - (1 - q(t))^3 - b(t)(1 - q(t))) \\ &\quad + b(t)(1 - q(t)) = p(t) \end{aligned}$$

Further Work

- $N \geq 4$
- General proof
- Establish bounds on the degrees of polynomials used in the polynomial matrix.