PERMUTATIONS AND THE DIVISOR GRAPH OF $[1, n]$

NATHAN MCNEW

ABSTRACT. Let $S_{\mathrm{div}}(n)$ denote the set of permutations π of n such that for each $1 \leq j \leq n$ either $j \mid \pi(j)$ or $\pi(j) \mid j$. These permutations can also be viewed as vertex-disjoint directed cycle covers of the divisor graph $\mathcal{D}_{[1,n]}$ on vertices $v_1, \ldots, v_n$ with an edge between v_i and v_j if $i \mid j$ or $j \mid i$. We improve on recent results of Pomerance by showing $c_d = \lim_{n \to \infty} (\#S_{\mathrm{div}}(n))^{1/n}$ exists and that $2.069 < c_d < 2.694$. We also obtain similar results for the set $S_{\mathrm{lcm}}(n)$ of permutations where $\mathrm{lcm}(j, \pi(j)) \leq n$ for all j. The results rely on a graph theoretic result bounding the number of vertex-disjoint directed cycle covers, which may be of independent interest.

1. INTRODUCTION

Several recent papers have investigated permutations π of the integers $[1, n] = \{1, 2, \ldots n\}$ where the value $\pi(i)$ is related to the index i in an arithmetically interesting way.

In [9] Pomerance introduces the *coprime permutations* (permutations satisfying the condition $\gcd(i, \pi(i)) = 1$ for all $1 \leq i \leq n$). He shows, for large n, that the count of such permutations is bounded between $n!/c_1^n$ and $n!/c_2^n$ with $c_1 = 3.73$ and $c_2 = 2.5$. That paper also includes a conjecture made by the author of this paper that the count of co-prime permutations of n, as $n \to \infty$, is $n!/c_o^{n+o(n)}$ where $c_o = \prod_p \frac{p(p-2)^{1-2/p}}{(p-1)^{2-2/p}} = 2.65044\ldots$. This conjecture was proven a few days later in a paper by Sah and Sawhney [12].

Coprime permutations are a special case of coprime mappings, bijective maps $f : A \to B$ between two sets of integers with the property that $\gcd(a, f(a)) = 1$ for every $a \in A$. Coprime mappings have been considered by many authors. Newman conjectured that there always exists a coprime mapping from $[1, n]$ to any set of n consecutive integers. This was proven by Pomerance and Selfridge [11]. The special case of coprime mappings from $[1, n]$ to $[n + 1, 2n]$ was proved in [3] (see also [2, 5]). More recently, coprime mappings have been shown to be related to the lonely runner conjecture [1, 8].

The same paper [9] that introduced the coprime permutations suggested other interesting arithmetic constraints that could be imposed on permutations. One example is the set $S_{\mathrm{div}}(n)$ of permutations π of n where, for each index i, either $i \mid \pi(i)$ or $\pi(i) \mid i$; another is the set $S_{\mathrm{lcm}}(n)$ of permutations where $\mathrm{lcm}(i, \pi(i)) \leq n$ for all i. (Notice that $S_{\mathrm{div}}(n) \subseteq S_{\mathrm{lcm}}(n)$.) A table with exact values of the counts of $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$ up to $n = 50$ is provided in Table 2 in the Appendix, along with the corresponding values of the n-th roots of these counts.

Pomerance revisits these two sets of permutations in [10]. In each case he shows that the count of these permutations grows geometrically. In particular, for large n,

$$1.9364^n \leq \#S_{\mathrm{div}}(n) < \#S_{\mathrm{lcm}}(n) \leq 13.5597^n.$$

A heuristic argument is given that $\#S_{\mathrm{lcm}}(n) \leq 8.3830^n$, though data suggests even this bound is substantially greater than the truth. He also gives the bounds $\#S_{\mathrm{lcm}}(n) \geq 2.1335^n$ and shows that $\#S_{\mathrm{lcm}}(n)/\#S_{\mathrm{div}}(n) > 1.00057^n$, both valid for large n.

In this paper, we show there exist constants c_d and c_l with $\#S_{\mathrm{div}}(n) = c_d^{n+o(n)}$ and $\#S_{\mathrm{lcm}}(n) = c_l^{n+o(n)}$ as n goes to infinity. The method immediately gives the improved upper bounds $c_d < 3.31369$ and $c_l \leq 6.60740$ without significant additional computation, but also gives an algorithm that can compute the constants to arbitrary precision, which we use to improve these upper bounds further in Section 6. Further ideas, discussed in Section 7, combined with the methods of [10] allow us to similarly improve the lower bounds, resulting in the bounds

$$2.06912 < c_d < 2.69390 \quad \text{and} \quad 2.30136 < c_l < 3.36352.$$

The key idea is a graph-theoretic result, bounding the ratio by which the count of the vertex-disjoint directed cycle covers of a graph can increase when a vertex is added, depending on the degree of that vertex. This result may also be of independent interest. A related question is posed in Section 8.

2. NOTATION AND STATEMENT OF RESULTS

It will be convenient to consider the permutations π in $S_{\mathrm{div}}(n)$ as vertex-disjoint directed cycle covers of the divisor graph $\mathcal{D}_{[1,n]}$ consisting of vertices $v_1, v_2, \ldots v_n$, and an edge between v_i and v_j if $i \mid j$ or $j \mid i$. Every vertex in the graph is treated as having a self-loop. In the context of this graph, permutations in $S_{\mathrm{div}}(n)$ are in bijection with vertex-disjoint directed cycle covers of $\mathcal{D}_{[1,n]}$. (The permutation $\pi \in S_{\mathrm{div}}(n)$ is associated with the cycle cover containing all directed edges of the form $i \to \pi(i)$.)

To account for fixed points and transpositions in the permutation, we allow cycles of any positive length in our cycle covers. In particular, cycles of length 1 (a single vertex involving only a loop of $\mathcal{D}_{[1,n]}$) and 2 (two vertices connected by an edge, in which case that edge is used two times) are allowed. In more generality, we write $\mathcal{D}_{[k,n]}$ for the divisor graph consisting of integers from the interval $[k,n]$, and $S_{\mathrm{div}}([k,n])$ for the permutations restricted to that interval.

In the same way, we let $\mathcal{L}_{[k,n]}$ be the graph consisting of integers from the interval $[k,n]$ with an edge between vertices v_i and v_j if $\mathrm{lcm}(i,j) \leq n$, and $S_{\mathrm{lcm}}([k,n])$ is the set of permutations subject to the same restriction. Then again we can view permutations in $S_{\mathrm{lcm}}(n)$ as vertex-disjoint directed cycle covers of $\mathcal{L}_{[1,n]}$.

For an arbitrary graph G, let $C(G)$ denote the number of vertex-disjoint directed cycle covers of the graph G. We will be interested in understanding how the count $G(G)$ grows as new vertices are added to the graph. For a graph G and a vertex v of G we define

$$R(G, v) := \frac{C(G)}{C(G - \{v\})}$$

to be the ratio of the number of vertex-disjoint directed cycle covers of G, divided by the number of such covers in the graph where the vertex v has been removed (or

alternatively cycle covers of G in which we insist that the vertex v be part of a cycle of length 1).

Intuitively, one would expect the ratio $R(G, v)$ to be larger for well connected vertices with high degree and smaller when the vertex v has small degree. It isn't obvious, however, that the ratio $R(G, v)$ would be bounded for all graphs G and vertices v having a fixed degree $d = d(v)$. This somewhat surprising result is given by the following theorem.

Theorem 2.1. *Let G be a graph with a self-loop on every vertex and v a vertex of G of degree $d = d(v)$, counting the self-loop once. Then*

$$R(G, v) \leq 1 + \frac{d^2 - d}{2}.$$

This theorem will be the main ingredient used to obtain a more precise estimate for $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$.

Theorem 2.2. *There exists an effectively computable constant $c_d = \lim\limits_{n \to \infty} (\#S_{\mathrm{div}}(n))^{1/n}$ and, for $\epsilon > 0$, we have*

$$\#S_{\mathrm{div}}(n) = c_d^{n\left(1 + O\left(\exp((-1+\epsilon)\sqrt{\log n \log \log n})\right)\right)}.$$

We obtain an identical result for $\#S_{\mathrm{lcm}}(n)$.

Theorem 2.3. *There exists an effectively computable constant $c_l = \lim\limits_{n \to \infty} (\#S_{\mathrm{lcm}}(n))^{1/n}$ and, for $\epsilon > 0$, we have*

$$\#S_{\mathrm{lcm}}(n) = c_l^{n\left(1 + O\left(\exp((-1+\epsilon)\sqrt{\log n \log \log n})\right)\right)}.$$

3. IMPROVED UPPER BOUNDS

Before proving these results, we can get a preview of how Theorem 2.1 will be used by applying it to get improved upper bounds for $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$. We start by writing $\#S_{\mathrm{div}}(n)$ as a telescoping product. This gives[1]

$$\#S_{\mathrm{div}}(n) = \prod_{a=1}^{n} \frac{\#S_{\mathrm{div}}([a, n])}{\#S_{\mathrm{div}}([a+1, n])} = \prod_{a=1}^{n} \frac{C(\mathcal{D}_{[a,n]})}{C(\mathcal{D}_{[a+1,n]})}$$

$$= \prod_{a=1}^{n} R(\mathcal{D}_{[a,n]}, v_a) \leq \prod_{a=1}^{n} \left(1 + \frac{d(v_a)^2 - d(v_a)}{2}\right)$$

using the bound from Theorem 2.1. Noting that the vertex v_a is connected to each of the $\lfloor \frac{n}{a} \rfloor$ multiples of a less than n (including itself) in $\mathcal{D}_{[a,n]}$ and subsequently

---

[1] Since the interval $[n+1, n]$ is the empty set, we set $\#S_{\mathrm{div}}([n+1, n]) = C(\mathcal{D}_{[n+1,n]}) = 1$.

grouping together vertices with the same degree we get

$$\#S_{\mathrm{div}}(n) \leq \prod_{a=1}^{n} \left(1 + \frac{d(v_a)^2 - d(v_a)}{2}\right) = \prod_{a=1}^{n} \left(1 + \frac{\left\lfloor \frac{n}{a}\right\rfloor^2 - \left\lfloor \frac{n}{a}\right\rfloor}{2}\right)$$

$$= \exp\left(\sum_{a=1}^{n} \log\left(1 + \frac{\left\lfloor \frac{n}{a}\right\rfloor^2 - \left\lfloor \frac{n}{a}\right\rfloor}{2}\right)\right)$$

$$= \exp\left(\sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k}\right\rfloor - \left\lfloor \frac{n}{k+1}\right\rfloor\right) \log\left(1 + \frac{k^2 - k}{2}\right)\right).$$

Finally, taking n-th roots and letting $n \to \infty$ we get that

$$\lim_{n\to\infty} (\#S_{\mathrm{div}}(n))^{1/n} \leq \lim_{n\to\infty} \exp\left(\frac{1}{n}\sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k}\right\rfloor - \left\lfloor \frac{n}{k+1}\right\rfloor\right) \log\left(1 + \frac{k^2 - k}{2}\right)\right)$$

$$= \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} \left(\frac{\log\left(1 + \frac{k^2-k}{2}\right)}{k(k+1)}\right) + O\left(\frac{\log n}{\sqrt{n}}\right)\right)$$

$$= \exp\left(\sum_{k=1}^{\infty} \frac{\log\left(1 + \frac{k^2-k}{2}\right)}{k(k+1)}\right) < \exp(1.19806) < 3.31369.$$

Note that when we remove the floor signs in the second line above, there are $\sqrt{n}$ values of $k \leq \sqrt{n}$, which each contribute $O(\log n)$ to the sum, and the difference $\left(\left\lfloor \frac{n}{k}\right\rfloor - \left\lfloor \frac{n}{k+1}\right\rfloor\right)$ is 0 for all but at most $\sqrt{n}$ values of $k > \sqrt{n}$, which again each contribute $O(\log n)$. This is how the error term in that line is obtained.

The numerical bound was obtained by evaluating the sum exactly for $1 \leq k \leq 10^7$ and then numerically bounding the contribution from the tail $k > 10^7$. We obtain further improved bounds below in Section 6.

Computing improved upper bounds for $\#S_{\mathrm{lcm}}(n)$ follows similarly, however there is some additional complexity, as the degree of v_a in $\mathcal{L}_{[a,n]}$ is not so simply described as in $\mathcal{D}_{[a,n]}$.

Proposition 3.1. *The degree of v_a in the graph $\mathcal{L}_{[a,n]}$ is*

$$d(v_a) = \sum_{1\leq j\leq \frac{n}{a}} \sum_{\substack{1\leq \ell\leq j \\ \gcd(j,\ell)=1 \\ \ell | a}} 1.$$

Proof. The vertex v_a is connected to v_b in $\mathcal{L}_{[a,n]}$ if $a \leq b \leq n$ and there are integers j, ℓ with $\mathrm{lcm}(a,b) = ja = \ell b \leq n$. In this case we must have $j \leq \frac{n}{a}$, $\ell \leq j$ (since $a \leq b$), $\gcd(j,\ell) = 1$, and so $\ell \mid a$. Having chosen values of a and j satisfying these conditions, $b = \frac{ja}{\ell}$ is uniquely determined and has $\mathrm{lcm}(a,b) \leq n$. Thus summing over all possible pairs ℓ and j gives the result. $\qquad\square$

Since

$$\sum_{\substack{1\leq \ell\leq j \\ \gcd(j,\ell)=1 \\ \ell | a}} 1 \leq \sum_{\substack{1\leq \ell\leq j \\ \gcd(j,\ell)=1}} 1 = \varphi(j),$$

we set $\Phi_k = \sum_{j=1}^{k} \varphi(j)$, and the proposition implies that the degree of v_a in $\mathcal{L}_{[a,n]}$ satisfies $d(v_a) \leq \Phi_{\lfloor \frac{n}{a} \rfloor}$. Following the same logic as for $\#S_{\mathrm{div}}(n)$, we get

$$\#S_{\mathrm{lcm}}(n) \leq \prod_{a=1}^{n} \left(1 + \frac{\Phi_{\lfloor \frac{n}{a} \rfloor}^2 - \Phi_{\lfloor \frac{n}{a} \rfloor}}{2} \right)$$

$$= \exp \left(\sum_{k=1}^{n} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n}{k+1} \right\rfloor \right) \log \left(1 + \frac{\Phi_k^2 - \Phi_k}{2} \right) \right).$$

So, taking n-th roots and letting $n \to \infty$ gives

$$\lim_{n \to \infty} \left(\#S_{\mathrm{lcm}}(n) \right)^{1/n} \leq \exp \left(\sum_{k=1}^{\infty} \frac{\log \left(1 + \frac{\Phi_k^2 - \Phi_k}{2} \right)}{k(k+1)} \right) \tag{1}$$

$$< \exp(1.88819) < 6.60740.$$

Again, the final bound was obtained by evaluating the sum exactly for $1 \leq k \leq 10^7$ and numerically bounding the contribution from the tail $k > 10^7$. We improve this bound further by handling the degrees of the vertices more carefully (and utilizing further numerical computations) in Section 6.

4. Graph Theory

In this section, we prove Theorem 2.1, the bound for the ratio $R(G, v)$ of the number of vertex-disjoint directed cycle covers of G including v to those with v removed (or fixed). The result is true for graphs in general, not just those obtained as divisor or lcm-graphs.

Suppose that v is a vertex of G of degree d (counting the loop on v.) Our goal is to show that

$$R(G, v) = \frac{C(G)}{C(G \setminus \{v\})} \leq 1 + \frac{d^2 - d}{2}.$$

Let $W = \{w_1, w_2, \ldots, w_{d-1}\}$ be the set of vertices adjacent to v in G, not including v itself. Write $C(G)$, the total number of vertex-disjoint directed cycle covers of G, as

$$C(G) = C_v + \sum_{i=1}^{d-1} C_{vw_i} + \sum_{\substack{1 \leq i,j < d \\ i \neq j}} C_{w_i v w_j}$$

where $C_v = C(G \setminus \{v\})$ is the number of vertex-disjoint directed cycle covers of G in which v is part of a cycle of length 1, C_{vw_i} is the count of those cycle covers in which v is part of a cycle of length 2 along with its neighbor w_i, and $C_{w_i v w_j}$ counts those where v is part of a cycle of length greater than 2 and contains the edges $w_i \to v \to w_j$.

The cycles counted by C_{vw_i} are in bijection with the of vertex-disjoint directed cycle covers of G in which both v and w_i are fixed (each part of cycles of length 1). Furthermore, the number of such cycles is less than C_v, the number of cycle covers where only v is required to be fixed, so $C_{vw_i} \leq C_v$.

In Lemma 4.1 below we consider $C_{w_i v w_j}$ and show, using an argument suggested by Petrov on Mathoverflow [7], that $\left(C_{w_i v w_j} \right)^2 \leq \frac{1}{4} \left(C_v \right)^2$, and thus $C_{w_i v w_j} \leq \frac{C_v}{2}$.

Inserting these inequalities $C_{vw_i} \leq C_v$ and $C_{w_i vw_j} \leq \frac{C_v}{2}$ into the definition of $R(G, v)$, we find that

$$R(G, v) = \frac{1}{C_v} \left(C_v + \sum_{i=1}^{d-1} C_{vw_i} + \sum_{\substack{1 \leq i,j < d \\ i \neq j}} C_{w_i vw_j} \right)$$

$$\leq 1 + (d-1) + (d-1)(d-2)\frac{1}{2} = 1 + \frac{d^2 - d}{2}$$

as desired.

It remains to demonstrate the bound mentioned above, comparing $C_{w_i vw_j}$ to C_v.

Lemma 4.1. *Fix distinct neighbors w_i and w_j of v in G and let C_v and $C_{w_i vw_j}$ be defined as above. Then*

$$\left(C_{w_i vw_j} \right)^2 \leq \frac{1}{4} \left(C_v \right)^2.$$

Proof. We first break up the count C_v, depending on whether or not each of the vertices w_i and w_j are fixed (a cycle of length 1) or part of a larger cycle. We write

$$C_v = X + Y_i + Y_j + Z$$

where X counts the contribution from the cases where v, w_i, and w_j are all part of 1-cycles, Y_i (respectively Y_j) counts those where only v and w_i (w_j) are fixed, and Z counts those where v is fixed but neither w_i nor w_j is fixed. We can then write

$$(C_v)^2 = (X + Y_i + Y_j + Z)^2 \geq (X + Z)^2 + (Y_i + Y_j)^2 \geq 4(XZ + Y_iY_j)$$

by the AM-GM inequality. So it remains to show that $XZ + Y_iY_j \geq (C_{w_i vw_j})^2$. We prove this by creating an injection from pairs of cycle-covers counted by the latter to those counted by the former.

Each of the products XZ, Y_iY_j and $(C_{w_i vw_j})^2$ count pairs of vertex-disjoint directed cycle covers. We color the one contributed by the first entry in the product blue, and the one from the second red. Drawing them together on the same graph results in a colored, directed multigraph in which every vertex has both an inward pointing blue and red edge, and a blue and a red edge directed away from it. Note that edges of G may be used multiple times. When this construction is done for a pair of coverings counted by $(C_{w_i vw_j})^2$, the result will contain both red and blue edges $w_i \to v \to w_j$.

We can now take this collection of red-and-blue-colored, directed edges and, instead of viewing it as two monochromatic vertex-disjoint directed cycle covers, we partition it into alternating-color directed cycles. There is a unique way to do this: at each vertex there is precisely one inward directed edge of each color and one outward directed edge of each color, so the cycles are found by alternately following blue and red edges. Every vertex is visited exactly twice by the resulting collection of alternating-color directed cycles. (A vertex may appear twice on the same cycle, or on two different cycles.)

The injection from objects counted by $(C_{w_i vw_j})^2$ to those counted by $XZ + Y_iY_j$ now proceeds as follows: Remove all four edges incident either to or from v. Then add two loops to v, one of each color. The result is a graph in which every vertex except for w_i and w_j has an inward and outward oriented edge of each color, while w_i has

two inward pointing edges of each color, and w_j has two outward pointing edges of each color. This graph can thus be decomposed into alternating-color directed cycles, as well as two alternating-color directed paths from w_j to w_i. Let P be the path that ends in a blue edge at w_i.

Now add a loop to each of the vertices w_i and w_j. While we can't initially assign these loops colors consistent with the coloring of the remainder of the edges (as each vertex is adjacent to precisely one blue and one red edge) we color the new loop on w_i blue. We then recolor every edge in the path P and reverse their directions, so that now the coloring and orientation of edges at w_i (and every intermediate vertex along P) is consistent. Since we have flipped the color and direction of one of the two edges adjacent to w_j, both edges now have the same color and opposite orientations. We can thus color the new loop at w_j the opposite color, resulting in a consistent coloring at that vertex as well.

The result is a new collection of alternating-color directed cycles of G, with v having two self-loops, and w_i and w_j each having a single loop. We can now separate this collection of edges back into two monochromatic vertex-disjoint directed cycle covers. We end up with two cases, based on whether the loops added to w_i and w_j are the same color or not. If they are both blue, then the red coloring fixes only v and neither w_i nor w_j, so this pair is counted by XZ. On the other hand, if the loops have opposite colors, then the resulting coloring is counted by Y_iY_j.

Every step of this construction is reversible, so it is easily checked that this is an inversion, completing the proof. (It is not a surjection—if the collection of alternating-color directed cycles produced from a pair of colorings counted by XZ or Y_iY_j does not result in both of the loops on w_i and w_j being part of the same alternating-color directed cycle, then that pair is not in the image of this map.) □

5. Asymptotic results for $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$

In [6], while investigating several counting problems related to the divisibility of integers in the interval $[1, n]$, the author proved a general result about functions $f(b, m)$ which *depend only on the connected component of v_b in the divisor graph of the interval $[b, n]$*. We say that a function $f : \mathbb{N}^2 \to \mathbb{R}$ has this property if $f(b, m) = f(b', m')$ whenever the connected component of v_b in $\mathcal{D}_{[b,m]}$ is isomorphic to the connected component of $v_{b'}$ in $\mathcal{D}_{[b',m']}$, with an isomorphism that maps v_b to $v_{b'}$. For functions with that property, [6] proves the following theorem:

Theorem 5.1 (Main Theorem of [6])**.** *Suppose $\epsilon > 0$, $A \geq 0$ and $f(a, n)$ is a bounded function $|f(a, n)| \leq A$ that depends only on the connected component of v_a in $\mathcal{D}_{[a,n]}$. Then there exists a constant*

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{1 \leq d \\ P^+(d) \leq i}} \sum_{t \in [id, (i+1)d)} \left(\frac{f(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right) \tag{2}$$

such that

$$\sum_{a=1}^{n} f(a, n) = nC_f + O_\epsilon \left(An \exp\left(-(1 - \epsilon)\sqrt{\log n \log \log n} \right) \right). \tag{3}$$

Here $P^+(d)$ denotes the largest prime divisor of d. The same result applies when divisor graphs are replaced by lcm graphs.

Letting $f(a,n) = \log\left(\frac{\#S_{\mathrm{div}}([a,n])}{\#S_{\mathrm{div}}([a+1,n])}\right)$, it is easy to see that the function f depends only on the connected component of v_a in the divisor graph. However, it does not satisfy the requirement that $|f(k,n)| \le A$ be bounded. For example, even in the case of $f(1,k)$, adding v_1 to the divisibility graph $\mathcal{D}_{[2,k]}$ connects all of the previously unconnected vertices v_p for prime integers $k/2 < p \le k$. Since they were previously unconnected, they were fixed by every permutation in $S_{\mathrm{div}}([2,k])$. In $S_{\mathrm{div}}([1,k])$ the integer 1 can be permuted with any of these primes, so we find that $f(1,k) > \log(\pi(k)-\pi(k/2))$, which goes to infinity with k. (Here $\pi(k)$ is the prime counting function.)

On the other hand, we can use Theorem 2.1 to show that even though $f(a,n)$ is not bounded, it cannot go to infinity very quickly as a function of the ratio $\frac{n}{a}$. In particular, we have

$$f(a,n) \le \log\left(1 + \frac{\left(\frac{n}{a}\right)^2 - \frac{n}{a}}{2}\right) = O\left(\log\frac{n}{a}\right).$$

In the expression for C_f and throughout the proof of Theorem 5.1 in [6], the variable i corresponds to this ratio $i = \left\lfloor\frac{a}{n}\right\rfloor$. If we inspect that proof carefully, replacing each instance that the bound $f(a,n) \le A$ is used with the bound $f(a,n) = O(\log i)$ instead, we find that the result is unchanged. (In the course of the proof the A that appears in each of the error terms ends up being replaced by a factor of $O(\log N)$, where N is a parameter in the proof describing where the sum over i is truncated. This factor of $\log N$ does not affect the optimization of the error terms, it remains optimal to take $\log N = \sqrt{\log n \log\log n}$. This additional factor can be absorbed into the existing final error term.) As a result, we can update the theorem to the form stated in Theorem 5.2 below.

Furthermore, it is straightforward to check that the proof in [6] applies equally well to the graphs $\mathcal{L}_{[a,n]}$. The key observation is that for any integers a and n, the set of vertices in the connected component of v_a in $\mathcal{D}_{[a,n]}$ is identical to the set of vertices in the connected component of v_a in $\mathcal{L}_{[a,n]}$, since if two vertices v_i and v_j are connected by an edge in $\mathcal{L}_{[a,n]}$, they are both connected to the vertex $v_{\mathrm{lcm}(i,j)}$ in $\mathcal{D}_{[a,n]}$. Combining these observations we obtain the following updated theorem.

Theorem 5.2. *Suppose $\epsilon > 0$, $f(a,n)$ depends only on the connected component of v_a in $\mathcal{D}_{[a,n]}$ (respectively $\mathcal{L}_{[a,n]}$) and $f(a,n) = O\left(\log\frac{n}{a}\right)$. Then, there exists a constant given by (2) such that the approximation (3) holds.*

Theorems 2.2 and 2.3 now follow as corollaries. We take

$$f(a,n) = \log\left(\frac{\#S_{\mathrm{div}}([a,n])}{\#S_{\mathrm{div}}([a+1,n])}\right) = \log\left(R(\mathcal{D}_{[a,n]}, v_a)\right)$$

(replacing $\mathcal{D}_{[a,n]}$ with $\mathcal{L}_{[a,n]}$ in the case of $\#S_{\mathrm{lcm}}(n)$), noting that the function depends only on the connected component of v_a in of the divisor graph $\mathcal{D}_{[a,n]}$ (or $\mathcal{L}_{[a,n]}$), and use Theorem 2.1 (as well as Proposition 3.1 in the case of $S_{\mathrm{lcm}}(n)$) to see that $f(a,n) = O\left(\log\frac{a}{n}\right)$ so that our updated theorem applies. Finally, as in Section 3, we

write $\#S_{\mathrm{div}}(n)$ as a telescoping product to get

$$\#S_{\mathrm{div}}(n) = \prod_{a=1}^{n} \frac{\#S_{\mathrm{div}}([a, n])}{\#S_{\mathrm{div}}([a+1, n])} = \exp\left(\sum_{a=1}^{n} f(a, n)\right),$$

or a similar statement for $\#S_{\mathrm{lcm}}(n)$, and the theorems follow from the estimate above.

6. NUMERICAL UPPER BOUNDS

The expressions given for the constants of Theorems 2.2 and 2.3 provide a method of computing the values of the constants to any precision by computing exactly sufficiently many terms and estimating the contribution for the remainder of the series. In practice, however, this is not such an easy task, as the values of $f(d, t)$ quickly become difficult to compute. Computing the number of vertex-disjoint directed cycle covers $C(G)$ of a graph G is equivalent to computing the permanent of the adjacency matrix of the graph, and computing matrix permanents is well known to be a computationally difficult problem

Nevertheless, computing exact values of $f(d, t)$ is tractable so long as the number of vertices in the corresponding graphs is not too large.

In order to improve the bound $c_d < 3.31369$ obtained in Section 3, we compute the exact value of $f(d, t)$ for the divisor graph for all i, d with $id < 50000$ and all $t \in [id, (i+1)d)$ for which the connected component of v_d in $\mathcal{D}_{[d,t]}$ had at most 50 vertices. For all of the uncomputed values of $f(d, t)$ we use the same bound $f(d, t) \leq \log\left(1 + \frac{i^2 - i}{2}\right)$, where $i = \lfloor \frac{t}{d} \rfloor$ as before. Doing so gives the improved upper bound $c_d < 2.69390$.

6.1. **Improved bounds for $\#S_{\mathrm{LCM}}(n)$.** There is more room for improvement in the upper bound $c_l < 6.60740$ obtained in (1). Before resorting to numerical calculation of the exact values of $f(d, t)$, as above, we can first get a significant improvement by more carefully dealing with the precise degree of vertices rather than using the crude bound $d(v_a) \leq \Phi_{\lfloor \frac{n}{a} \rfloor}$ as we did in Section 3.

In particular, Proposition 3.1 tells us that the degree of v_a in $\mathcal{D}_{[a,n]}$ depends on both $i = \lfloor \frac{n}{a} \rfloor$, and the set of integers $\{\ell < i : \ell \mid a\}$. This set is completely characterized by the integer $d' = \gcd(a, M_i)$, where $M_i := \mathrm{lcm}([1, i])$ is the least common multiple of the first i integers. Any integer a in the interval $\left(\frac{n}{i+1}, \frac{n}{i}\right]$ having $\gcd(a, M_i) = d'$ will have the same degree, which we denote $T_{i,d'}$. Namely, from Propositon 3.1,

$$T_{i,d'} := d(v_a) = \sum_{\substack{1 \leq j \leq i}} \sum_{\substack{1 \leq \ell \leq j \\ \gcd(j,\ell)=1 \\ \ell \mid d'}} 1.$$

For a fixed $d' \mid M_i$, the proportion of integers a having $d' = \gcd(a, M_i)$ is $\frac{\varphi\left(\frac{M_i}{d}\right)}{M_i}$. Thus we can improve the upper bound 1 to

$$\lim_{n \to \infty} (\#S_{\mathrm{lcm}}(n))^{1/n} \leq \exp\left(\sum_{i=1}^{\infty} \sum_{d' \mid M_i} \left(\frac{\varphi\left(\frac{M_i}{d}\right)}{i(i+1)M_i} \log\left(1 + \frac{T_{i,d'}^2 - T_{i,d'}}{2}\right)\right)\right).$$

We first numerically compute the value of this sum for $1 \leq i \leq 10000$. Computing the inner sum over $d' \mid M_i$ quickly becomes unwieldy as i gets large, however. So

for $i \geq 42$ we only compute it over the restricted range $d' \leq 10000$. For all of the uncomputed values we use the same bound $T_{i,d'} \leq \Phi_i$ as before. We similarly use this bound for $10000 < i \leq 10^7$, and numerically estimate the tail $i > 10^7$, giving the bound $c_l \leq 4.25724$.

Finally, we can numerically compute exact values of $f(d, t)$, as we did in the divisor graph case above. Doing so for pairs (i, d) with $id < 50000$ and all $t \in [id, (i+1)d)$ for which the connected component of v_d in $\mathcal{L}_{[d,t]}$ had at most 44 vertices[2] gives the improved bound $c_l \leq 3.36352$.

To compute matrix permanents, a parallel algorithm [4], optimized for sparse binary matrices, was used. The code provided by the author of that paper was modified to use exact large integer arithmetic instead of floating point arithmetic.

7. NUMERICAL LOWER BOUNDS

As mentioned above, Theorems 2.2 and 2.3 give a method to compute lower bounds for c_d and c_l by explicitly computing partial sums of the series given for each constant, but the convergence is quite slow. Using the same range of values used to obtain the upper bounds in the previous section (all pairs (i, d) with $id < 50000$ and all $t \in [id, (i+1)d)$ for which the connected component of v_d in $\mathcal{D}_{[d,t]}$ had at most 50 vertices (or in the case of c_l, the connected component of $\mathcal{L}_{[d,t]}$ had at most 44 vertices) gives lower bounds $1.70584 < c_d$, and $1.81576 < c_l$, both substantially worse than the bounds $1.9364 < c_d$ and $2.1335 < c_l$ obtained by Pomerance in [10].

Pomerance obtains his bounds as follows. For an integer b, write the divisors $a_i \mid b$, as $1 = a_1 < a_2 < \cdots a_k = b$, where $k = \tau(b)$. Write $S_{\mathrm{div}}(\{a_1, a_2 \ldots a_i\})$ for the permutations of the set $\{a_1, a_2 \ldots a_i\}$ satisfying the usual divisibility requirements. (Similarly for $S_{\mathrm{lcm}}(\{a_1, a_2 \ldots a_i\})$ in which case we impose the condition $\mathrm{lcm}(a_j, \pi(a_j)) \leq a_i$, where a_i is the largest element in the set.) Then define

$$\beta_{\mathrm{div}}(b) := \frac{\log(\#S_{\mathrm{div}}(\{a_1, a_2, \ldots, a_k\}))}{b} + \sum_{i=1}^{k-1} \left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right) \log(\#S_{\mathrm{div}}(\{a_1, a_2, \ldots, a_i\}))$$

and

$$\beta_{\mathrm{lcm}}(b) := \frac{\log(\#S_{\mathrm{lcm}}(\{a_1, a_2, \ldots, a_k\}))}{b} + \sum_{i=1}^{k-1} \left(\frac{1}{a_i} - \frac{1}{a_{i+1}}\right) \log(\#S_{\mathrm{lcm}}(\{a_1, a_2, \ldots, a_i\})).$$

Note that $\#S_{\mathrm{lcm}}(\{a_1, a_2, \ldots, a_k\}) = k!$, since $\mathrm{lcm}(a_i, a_j) \leq a_k = b$ for all i, j. Finally, define $\alpha(p^i)$ on prime powers by

$$\alpha(p^i) = \frac{p^{i+1} - p^i}{p^{i+1} - 1}$$

and extend α to a multiplicative function on the positive integers. With this definition, $\alpha(b)$ gives the density of the integers j satisfying the condition $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$ for every prime p. Pomerance then shows that for any integer b (as $n \to \infty$) both

$$\#S_{\mathrm{div}}(n) \geq \exp\left(\beta_{\mathrm{div}}(b)\alpha(b)n + o(n)\right) \tag{4}$$

---

[2]Computing matrix permanents was slower in this case because the matrices were more dense, which necessitated a smaller maximum matrix size.

and

$$\#S_{\mathrm{lcm}}(n) \geq \exp\left(\beta_{\mathrm{lcm}}(b)\alpha(b)n + o(n)\right). \tag{5}$$

b	$\beta_{\mathrm{div}}(b)\alpha(b)$	$e^{\beta_{\mathrm{div}}(b)\alpha(b)}$	$\beta_{\mathrm{lcm}}(b)\alpha(b)$	$e^{\beta_{\mathrm{lcm}}(b)\alpha(b)}$
24	0.542689	1.720627	0.602065	1.825886
48	0.578121	1.782686	0.638299	1.893259
60	0.646855	1.909527	0.646855	1.909527
72	0.598295	1.819015	0.670619	1.955447
120	0.610358	1.841091	0.707611	2.029138
144	0.631752	1.880903	0.704928	2.023701
180	0.710735	2.035488	0.710735	2.035488
240	0.642829	1.901853	0.740126	2.096201
288	0.650370	1.916251	0.723606	2.061856
360	0.660646	1.936043	0.769250	2.158148
432	0.654328	1.923851	0.730695	2.076524
480	0.660864	1.936465	0.757764	2.133502
576	0.660597	1.935949	0.733726	2.082827
720	0.691601	1.996910	0.800104	2.225772
864	0.672306	1.958749	0.748644	2.114132
1440	0.708710	2.031369	0.816672	2.262957
1728	0.682146	1.978118	0.758293	2.134630
2160	0.711797	2.037650	0.822675	2.276582
3456	0.687472	1.988683	0.763452	2.145671
5184	0.690644	1.995001	0.767521	2.154419
10368	0.695844	2.005402	0.772521	2.165219

TABLE 1. Values of $\beta_{\mathrm{div}}(b)\alpha(b)$ and $\beta_{\mathrm{lcm}}(b)\alpha(b)$ computed for 3- and 5-smooth values of b.

The bounds above are obtained by taking $b = 480$. We can first improve the bounds by extending the computations to larger values of b as given in Table 1. By using the values obtained for $b = 2160 = 2^4 \times 3^3 \times 5$, we can improve the lower bounds to $2.037650 < c_d$ and $2.276582 < c_l$.

We can further improve these bounds by combining these ideas (for small primes) with exact computations for the large primes. An inspection of the proof in [10] shows that (4) and (5) are in fact lower bounds for "smooth" versions of $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$ in the following sense.

Let $S_{\mathrm{div}}(n, s) \subseteq S_{\mathrm{div}}(n)$ be the set of permutations π of n such that for each i, either $i \mid \pi(i)$, or $\pi(i) \mid i$ and, furthermore, the quotient is s-smooth (meaning that $P^+\left(\frac{i}{\pi(i)}\right) \leq s$ or $P^+\left(\frac{\pi(i)}{i}\right) \leq s$, whichever is an integer). In the same way, let $S_{\mathrm{lcm}}(n, s) \subseteq S_{\mathrm{lcm}}(n)$ be the permutations of n where for each i, $\mathrm{lcm}(i, \pi(i)) \leq n$ and also $P^+\left(\frac{\mathrm{lcm}(i, \pi(i))}{\gcd(i, \pi(i))}\right) \leq s$. Then we have

$$\#S_{\mathrm{div}}(n, P^+(b)) \geq \exp\left(\beta_{\mathrm{div}}(b)\alpha(b)n + o(n)\right) \tag{6}$$

and

$$\#S_{\mathrm{lcm}}(n, P^+(b)) \geq \exp\left(\beta_{\mathrm{lcm}}(b)\alpha(b)n + o(n)\right).$$

We can then improve the lower bounds for $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$ by obtaining lower bounds for each of the ratios $\frac{\#S_{\mathrm{div}}(n)}{\#S_{\mathrm{div}}(n,s)}$ and $\frac{\#S_{\mathrm{lcm}}(n)}{\#S_{\mathrm{lcm}}(n,s)}$. This can be done through

explicit computation, similar to the methods introduced in previous sections. Fix a smoothness bound s and let $\mathcal{R}_s \subset [1, n]$ be the set of s-rough integers up to n, those integers free of prime factors less than or equal to s. Then, for an integer $r \in \mathcal{R}_s$ let $M_{r,n,s} = \{i \le n : r \mid i \text{ and } P^+(\frac{i}{r}) \le s\}$ be the set of s-smooth multiples of r up to n. Note that we can then write

$$\#S_{\text{div}}(n, s) = \prod_{r \in \mathcal{R}_s} \#S_{\text{div}}(M_{r,n,s})$$

since permutations in $S_{\text{div}}(n, s)$ only permute integers within the sets $M_{r,n,s}$.

In a similar way, if $A_1, A_2, \ldots A_k$ are any disjoint subsets of $[1, n]$, then

$$\#S_{\text{div}}(n) \ge \prod_{i=1}^{k} \#S_{\text{div}}(A_i).$$

Combining these two observations, if $D_1, D_2, \ldots, D_k$ are any disjoint sets that partition $\mathcal{R}_s$ then

$$\frac{\#S_{\text{div}}(n)}{\#S_{\text{div}}(n, s)} = \frac{\#S_{\text{div}}(n)}{\prod_{i=1}^{k} \prod_{r \in D_i} \#S_{\text{div}}(M_{r,n,s})} \ge \prod_{i=1}^{k} \frac{\#S_{\text{div}}\left(\bigcup_{r \in D_i} M_{r,n,s}\right)}{\prod_{r \in D_i} \#S_{\text{div}}(M_{r,n,s})} \tag{7}$$

since each of the sets $M_{r,n,s}$ are disjoint and partition the interval $[1, n]$. Furthermore, each of the terms in the rightmost product above are greater than or equal to 1 since the permutations enumerated by the denominator are a subset of those counted by the numerator. So, multiplying over a set of disjoint D_i which don't necessarily partition $\mathcal{R}_s$, we still obtain a lower bound for the ratio.

We now use the following strategy to pick sets D_i. Consider $\mathcal{D}_{\mathcal{R}_s}$ the restriction of the divisor graph to the set $\mathcal{R}_s$. For an integer $a \in \mathcal{R}_s$, take $D_{a,n}$ to be the integers in the connected component of v_a in $\mathcal{D}_{\mathcal{R}_s \cap [a,n]}$ (further restricting to integers greater than or equal to a). Now we take A_s to be the set of integers $a \in \mathcal{R}_s$ satisfying:

- $|\bigcup_{r \in D_{a,n}} M_{r,n,s}| \le 50$
- If $i = \lfloor \frac{n}{a} \rfloor$ and d is the largest i-smooth divisor of a, then $id < 50000$.
- $D_{a,n}$ is disjoint from any $D_{b,n}$ with $b < a$ that satisfies the first two properties.

We note that in this setup with $i = \lfloor \frac{n}{a} \rfloor$, d the largest i-smooth divisor of a and $t = \lfloor \frac{nd}{a} \rfloor$, the set $\bigcup_{r \in D_{a,n}} M_{r,n,s}$ has the same exact multiplicative structure as $\bigcup_{r \in D_{d,t}} M_{r,t,s}$. So, using the same techniques as previous sections, we can compute

$$\lim_{n \to \infty} \left(\prod_{a \in A_S} \frac{\#S_{\text{div}}\left(\bigcup_{r \in D_a} M_{r,n,s}\right)}{\prod_{r \in D_{d,t}} \#S_{\text{div}}(M_{r,n,s})} \right)^{1/n}$$

$$= \prod_{\substack{i,d,t \\ 1 \le d \le \frac{50000}{i} \\ d \in \mathcal{R}_s,\ P^+(d) \le i \\ t \in [id, (i+1)d) \\ \left|\bigcup_{r \in D_{d,t}} M_{r,t,s}\right| \le 50}} \left(\frac{\#S_{\text{div}}\left(\bigcup_{r \in D_{d,t}} M_{r,t,s}\right)}{\prod_{r \in D_{d,t}} \#S_{\text{div}}(M_{r,t,s})} \middle/ \frac{\#S_{\text{div}}\left(\bigcup_{\substack{r \in D_{d,t} \\ r > d}} M_{r,t,s}\right)}{\prod_{\substack{r \in D_{d,t} \\ r > d}} \#S_{\text{div}}(M_{r,t,s})} \right)^{\frac{1}{t(t+1)} \prod_{p \le i} \frac{p-1}{p}}. \tag{8}$$

Setting $s = 3$ we can evaluate the expression above exactly and find that it evaluates to $0.031282\ldots$. Using this value in (7), along with the bound for $\#S_{\mathrm{div}}(n, 3)$ obtained using $b = 10368$ (see Table 1) we compute that

$$c_d \geq \exp(0.695844 + 0.031282) = 2.06912.$$

We can also evaluate the expression in (8) when $s = 5$, in which case we obtain the value $0.00905\ldots$. However using this, along with the value of $\beta_{\mathrm{div}}(b)\alpha(b)$ for $b = 2160$, the largest 5-smooth b-value we were able to compute, does not result in a better lower bound for c_d.

Following the same strategy for $\#S_{\mathrm{lcm}}(n)$, we take L_b to be the integers in the connected component of v_b in $\mathcal{L}_{\mathcal{R}_s \cap [b,n]}$, and B_s the set of integers $b \in \mathcal{R}_s$ satisfying analogous conditions as A_s, but with the size restricted to 44 instead of 50 in the first condition. We then compute

$$\lim_{n \to \infty} \left(\prod_{a \in B_3} \frac{\#S_{\mathrm{lcm}}\left(\bigcup_{d \in L_a} M_{d,n,s}\right)}{\prod_{d \in L_a} \#S_{\mathrm{lcm}}(M_{d,n,s})} \right)^{1/n} = 0.040298\ldots,$$

$$\lim_{n \to \infty} \left(\prod_{a \in B_5} \frac{\#S_{\mathrm{lcm}}\left(\bigcup_{d \in L_a} M_{d,n,s}\right)}{\prod_{d \in L_a} \#S_{\mathrm{lcm}}(M_{d,n,s})} \right)^{1/n} = 0.010828\ldots.$$

Comparing these to the values in Table 1, we find that this time around the best lower bound is obtained using $s = 5$ and $b = 2160$, in which case we obtain

$$c_l \geq \exp(0.822675 + 0.010828) > 2.30136.$$

It isn't clear whether these methods can be used to improve the lower bound $c_\ell/c_d > 1.00057$ obtained in [10].

Question 7.1. *Can the lower bound for c_ℓ/c_d be improved?*

8. Final remarks on Theorem 2.1

The bound obtained in Theorem 2.1 does not seem to be sharp for any $d > 2$. For example, among all graphs G of order at most 10 and vertices $v \in G$ of degree 3 (counting the loop on v), the bound $R(G, v) \leq 3$ holds (compared to the bound $R(G, v) \leq 4$ given by the theorem). $R(G, v) = 3$ when $G = K_3$, with loops added to each vertex. Among graphs of order at most 10 and vertices of degree 4 we have $R(G, v) \leq \frac{19}{4}$ (compared to 7), achieved when $G = K_{2,3}$ with loops added to each vertex (and v being one of the two vertices of degree 4).

The statement of Theorem 2.1 is reminiscent of Bregman's inequality, which states, in the notation of this paper, that

$$C(G) \leq \prod_{v \in V(G)} (d(v)!)^{1/d(v)}.$$

This comparison might lead one to conjecture that $R(G, v) = O(d(v))$, however this is false: Fixing $d > k > 1$ and letting v be one of the k vertices of degree $d + 1$ in the

complete bipartite graph $K_{d,k}$ (with loops added to each vertex), we find that

$$R(K_{d,k}, v) = \frac{C(K_{d,k})}{C(K_{d,k-1})} = \frac{\sum_{i=0}^{k} \binom{k}{i} \frac{d!(k-i)!}{(d-k+i)!}}{\sum_{i=0}^{k-1} \binom{k-1}{i} \frac{d!(k-1-i)!}{(d-k+i+1)!}}.$$

Taking $k = \frac{d}{2}$, we find that the expression above simplifies to $\frac{d^2}{4} + O(d)$. So the bound in Theorem 2.1 has the optimal order of growth, though it is not clear what the optimal constant in such a bound would be.

Question 8.1. *What is the best possible constant c in a bound of the form $R(G, v) \leq cd^2(1 + o(1))$ as $d(v) \to \infty$?*

From the example above and Theorem 2.1, we know that the optimal constant is between $\frac{1}{4}$ and $\frac{1}{2}$. It would also be of interest to determine, for fixed degrees d, the best upper bound of this sort.

Acknowledgments

References

[1] T. Bohman and F. Peng, *Coprime mappings and lonely runners*, Mathematika **68** (2022), no. 3, 784–804. MR4449832

[2] V. Chvátal, *A remark on Newman's conjecture*, Proceedings of the Washington State University Conference on Number Theory (Washington State Univ., Pullman, Wash., 1971), 1971, pp. 113–129. MR0319870

[3] D. E. Daykin and M. J. Baines, *Coprime mappings between sets of consecutive integers*, Mathematika **10** (1963), 132–136. MR0163877

[4] K. Kaya, *Parallel algorithms for computing sparse matrix permanents*, Turkish Journal of Electrical Engineering and Computer Sciences **27** (2019), no. 6, 4284–4297.

[5] B. Larsen, H. Lehmann, A. Park, and L. Robertson, *Coprime mappings on n-sets*, Integers **17** (2017), Paper No. A51, 11. MR3724321

[6] N. McNew, *Counting primitive subsets and other statistics of the divisor graph of $\{1, 2, \ldots, n\}$*, European J. Combin. **92** (2021), Paper No. 103237, 20. MR4149161

[7] F. Petrov, *A bound for the permanent of a nonnegative matrix*, MathOverflow. https://mathoverflow.net/q/424593.

[8] C. Pomerance, *Coprime matchings*, Integers **22** (2022), Paper No. A2, 9. MR4363102

[9] _____, *Coprime permutations*, Integers **22** (2022), Paper No. A83, 20.

[10] _____, *Permutations with arithmetic constraints*, arXiv preprint (2022), available at arXiv:2206.01699.

[11] C. Pomerance and J. L. Selfridge, *Proof of D. J. Newman's coprime mapping conjecture*, Mathematika **27** (1980), no. 1, 69–83. MR581998

[12] A. Sah and M. Sawhney, *Enumerating coprime permutations*, Mathematika **68** (2022), no. 4, 1120–1134.

APPENDIX A. TABLE OF VALUES OF $\#S_{\mathrm{div}}(n)$ AND $\#S_{\mathrm{lcm}}(n)$

n	$\#S_{\mathrm{div}}(n)$	$(\#S_{\mathrm{div}}(n))^{1/n}$	$\#S_{\mathrm{lcm}}(n)$	$(\#S_{\mathrm{lcm}}(n))^{1/n}$
1	1	1.000000	1	1.000000
2	2	1.414214	2	1.414214
3	3	1.442250	3	1.442250
4	8	1.681793	8	1.681793
5	10	1.584893	10	1.584893
6	36	1.817121	56	1.955981
7	41	1.699799	64	1.811447
8	132	1.841076	192	1.929357
9	250	1.846876	332	1.906016
10	700	1.925351	1184	2.029248
11	750	1.825447	1264	1.914155
12	4010	1.996467	12192	2.190313
13	4237	1.901098	12872	2.070745
14	10680	1.939792	37568	2.122134
15	24679	1.962555	100836	2.155631
16	87328	2.036208	311760	2.204772
17	90478	1.956867	322320	2.108710
18	435812	2.057285	2338368	2.258544
19	449586	1.983885	2408848	2.167129
20	1939684	2.062465	14433408	2.280176
21	3853278	2.058785	32058912	2.277331
22	8650900	2.066907	76931008	2.282754
23	8840110	2.004564	78528704	2.204256
24	60035322	2.109115	919469408	2.363092
25	80605209	2.071355	1158792224	2.304413
26	177211024	2.076107	2689828672	2.305057
27	368759752	2.076284	4675217824	2.281082
28	1380348224	2.120451	21679173184	2.339615
29	1401414640	2.067278	21984820864	2.273133
30	8892787136	2.146024	381078324992	2.432393
31	9014369784	2.094725	386159441600	2.364649
32	33923638848	2.133429	1202247415040	2.385063
33	59455553072	2.120756	2207841138624	2.366245
34	126536289568	2.120976	4860086689536	2.361221
35	207587882368	2.105468	8681783534848	2.342473
36	1495526775088	2.178656	112777175188224	2.456628
37	1510769105288	2.133868	113878087417856	2.398302
38	3187980614208	2.133241	247857779387904	2.392185
39	5415462995568	2.120818	437979951107072	2.373679
40	29811240618112	2.171998	3191130554148864	2.441173
41	30071845984896	2.131747	3217753817425920	2.389093
42	167426899579520	2.181034	40769431338324480	2.485903
43	168778036632608	2.142238	41092863780506112	2.434258
44	543720217208896	2.162195	148296367650710016	2.456139
45	1741288345700048	2.181152	512674391975854336	2.474868
46	3618889806595872	2.178863	1089866926717622272	2.466703
47	3643985571635136	2.143371	1097208951955834368	2.420115
48	28167109438114448	2.201420	13688883937198881792	2.504231
49	33158989380172192	2.173477	15603885890357429248	2.464329
50	107833432035711440	2.191064	58892187478016638976	2.485428

TABLE 2. Values of $\#S_{\mathrm{div}}(n)$ and $\#S_{\mathrm{lcm}}(n)$ up to $n = 50$.

DEPARTMENT OF MATHEMATICS, TOWSON UNIVERSITY, TOWSON, MD 21252, USA
Email address: nmcnew@towson.edu