EXPLICIT BOUNDS FOR LARGE GAPS BETWEEN SQUAREFREE AND CUBEFREE INTEGERS

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Abstract. We obtain explicit forms of the current best known asymptotic upper bounds for gaps between squarefree and cubefree integers. In particular we show, for any $x \geq 2$, that every interval of the form $(x, x + 11x^{1/5}\log x]$ contains a squarefree integer and every interval $(x, x + 5x^{1/7}\log x]$ contains a cubefree integer. The constants 11 and 5 can be improved further, if x is assumed to be larger than a very large constant.

1. Introduction

An integer n is called *squarefree* if it is not divisible by the square of any prime p. More generally, if $k \geq 2$, n is called *k-free* if it is not divisible by p^k for any prime p. Just as we commonly refer to 2-free integers as squarefree, 3-free integers are also known as *cubefree*.

The asymptotic distribution of the k-free integers has been studied systematically, at least since the early 1900s, with a special focus on the squarefree case. Let $Q_k(x)$ denote the counting function of the k-free numbers up to x, and consider the error term $E_k(x)$ in the asymptotic formula

$$Q_k(x) = \frac{x}{\zeta(k)} + E_k(x),$$

where $\zeta(k)$ is the Riemann zeta-function. The bound $E_k(x) = O(x^{1/k})$ is classical, and further improvements are closely related to the distribution of zeros of the zeta-function. In particular, the best known bound for $E_k(x)$,

$$E_k(x) = O\left(x^{1/k}\exp\left(-c(k)(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right),$$

follows from the work of Walfisz on the error term in the Prime Number Theorem (see [32]). Still, a number of authors [1, 2, 11, 18–20, 22] have obtained sharper bounds under the assumption that the Riemann Hypothesis is true.

A related problem that has attracted considerable attention concerns the gaps between consecutive k-free integers. The first result in that direction was obtained by Fogels [10], who proved that if $\theta > 2/5$ the interval $(x, x + x^\theta]$ contains a squarefree integer for all sufficiently large x. In 1951, Roth [28] reduced the exponent 2/5 in Fogels's result to 3/13, while Halberstam and Roth [14] proved that the interval $(x, x + x^\theta]$ contains a k-free integer for any $\theta > 1/(2k)$ and for all sufficiently large x.

Around the same time, Erdős [3] proved that there exist infinitely many intervals $(x, x+h]$, with

$$h \gg \frac{\log x}{\log\log x},$$

which contain no squarefree integers. Together, these results inspired the conjecture that for any fixed $\varepsilon > 0$, the interval $(x, x+x^{\varepsilon}]$ contains a squarefree integer for sufficiently large x. This conjecture seems beyond the reach of current methods, though Granville [13] has shown that, like many other famous theorems and conjectures in number theory, it follows from the *abc*-conjecture of Masser and Oesterlé.

Initially, further improvements on Roth's result [28] on gaps between squarefree numbers were obtained through the method of exponential sums [12, 25, 26, 29], while the (mostly elementary) work of Halberstam and Roth [14] inspired research on the distribution of k-free numbers in polynomial sequences: see [15, 16, 24] for some early work and [6, §2] for a more detailed history. Starting in the late 1980s, Filaseta and Trifonov published a series of papers [4, 5, 7–9, 30, 31], where they developed an elementary proof [8] that there exists a constant $c > 0$ such that the interval $(x, x + cx^{1/5}\log x]$ contains a squarefree integer for all sufficiently large x. Later, Trifonov [31] generalized this result and proved that, for each $k \geq 3$, there exists a constant $c = c(k) > 0$ such that the interval $(x, x+cx^{1/(2k+1)}\log x]$ contains a k-free integer for all sufficiently large x. Filaseta and Trifonov [9] generalized their method to achieve progress in other problems—see the survey article [6] for the history of such developments, but sharper bounds on the gaps between k-free integers have remained elusive.

During the past couple of decades, number theorists' interest in numerically explicit results has increased significantly, and this has led to the development of numerically explicit versions of known theorems. As the Filaseta–Trifonov approach to gaps between k-free integers is both self-contained and "numerically friendly," it therefore makes sense to investigate fully explicit versions of the results of [8] and [31]. In this note, we prove such explicit versions of the gap results for squarefree and cubefree integers. Our two main theorems are as follows.

Theorem 1. *For any $x \geq 2$, the interval $(x, x+11x^{1/5}\log x]$ contains a squarefree integer.*

Theorem 2. *For any $x \geq 2$, the interval $(x, x+5x^{1/7}\log x]$ contains a cubefree integer.*

The focus of the above theorems is on providing explicit intervals that work for all x. The price we pay for this universality are the somewhat elevated values of the constants 11 and 5 in the theorems. If one is interested in reducing those constants further and willing to accept a result that holds only for sufficiently large x, then one may prefer the versions given in the next two theorems.

Theorem 3. *Every interval*

- $(x, x + 5x^{1/5} \log x]$ *contains a squarefree number for* $x \geq e^{400}$*;*
- $(x, x + 2x^{1/5} \log x]$ *contains a squarefree number for* $x \geq e^{1800}$*;*
- $(x, x + x^{1/5} \log x]$ *contains a squarefree number for* $x \geq e^{500\,000}$*.*

Theorem 4. *Every interval*

- $(x, x + 2x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{550}$*;*
- $(x, x + x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{2300}$*;*
- $(x, x + \frac{1}{2}x^{1/7} \log x]$ *contains a cubefree number for* $x \geq e^{75\,000}$*.*

Mossinghoff, Oliveira e Silva and Trudgian [23] (see also Marmet [21]) investigated long gaps between squarefree numbers numerically. Their computational work establishes the size of the longest gaps up to 10^{18}, which are all dramatically smaller than the bounds that we get in this paper. The largest gap that they find is a string of 18 consecutive non-squarefree numbers, the first of which is 125 781 000 834 058 568. As a result of their work, we can assume $x \geq 10^{18} > e^{41}$ throughout the rest of this paper.

Theorems 3 and 4 already hint that the constants in the two main theorems are influenced by the "small" values of x. Indeed, we establish Theorems 1 and 2 for $x \geq e^{116}$ and $x \geq e^{210}$, respectively. To bridge the gap between those lower bounds and e^{41}, we prove several propositions giving results with larger exponents, which are however superior to the results of the theorems for small x. In particular, we find that the interval $(x, x + 5x^{1/4}]$ always contains a squarefree integer (Proposition 3) and the interval $(x, x + 3.8x^{1/4}]$ contains a squarefree integer for $x \geq e^{109}$ (Proposition 4). In the cubefree case, we show that the interval $(x, x + 2x^{1/5}]$ always contains a cubefree integer (Proposition 5) and the intervals $(x, x + 10x^{1/6}]$ and $(x, x + 8.5x^{1/6}]$ contain a cubefree integer for $x \geq e^{95}$ and $x \geq e^{191}$ respectively (Proposition 6).

It should be clear by now from the above discussion, that the values of the constants and the varios cutoffs in the theorems (and in Propositions 3–6) are not exact, but rather "nice" approximations. We say more about this in Section 8.[1]

Finally, we should point out that the strategies we apply here to deal with squarefree and cubefree integers can be used to get explicit bounds for gaps between k-free integers for arbitrary k as well. However, since additional work is required to generalize many of the polynomial identities used, we explore this in a future paper.

Notation. Throughout the paper, for a real number θ, we use $\lfloor \theta \rfloor$ to denote the greatest integer less than or equal to θ; also, $\{\theta\} = \theta - \lfloor \theta \rfloor$. We write $|A|$ for the size of the set A, and $\pi(x)$ for the prime counting function. Finally, we use $c_1, c_2, \ldots$ to denote constants that appear in the proofs. Those constants tend to

[1]The interested reader can explore these phenomena further using the `SageMath` code for the computational part of our work, which is available at https://github.com/agreatnate/explicit-k-free-integer-bounds

depend on various parameters introduced throughout our arguments (such as λ, δ and m); we may indicate such dependencies by labeling our constants as functions of said parameters—see the constant $c_2(m)$ in (3.15), for example.

2. Preliminaries

2.1. **Outline of the method.**

For $k \geq 2$, let $N_k(x,h)$ be the number of integers in $(x, x+h]$ that are not k-free. Clearly, to prove any of our theorems, it suffices to show that $N_k(x,h) < h-1$ for the respective choices of k, x and h. We first sieve this interval of the squares of very small primes, up to a parameter J to be chosen later. The number of integers in $(x, x+h]$ divisible by the k-th power of a prime up to J is at most

$$h\left(1 - \prod_{p\leq J}\left(1-\frac{1}{p^k}\right)\right) + 2^{\pi(J)} = h\left(1 - \prod_{p\leq J}\left(1-\frac{1}{p^k}\right) + \frac{2^{\pi(J)}}{h}\right) =: h\sigma_0'(h,J).$$

We then count separately the integers divisible by p^k for each prime $p > J$. We find that

$$N_k(x,h) \leq h\sigma_0'(h,J) + \sum_{p>J}\left(\left\lfloor\frac{x+h}{p^k}\right\rfloor - \left\lfloor\frac{x}{p^k}\right\rfloor\right), \tag{2.1}$$

where the sum on the right is over all primes greater than J. To bound the latter sum, we study separately the contributions of "small" and "large" primes p. We introduce a parameter H, which we will later choose as $H = mh$, with $m \geq 1$ of moderate size, and we use this parameter to split the sum in (2.1) as follows:

$$\left(\sum_{J<p\leq H} + \sum_{p>H}\right)\left(\left\lfloor\frac{x+h}{p^k}\right\rfloor - \left\lfloor\frac{x}{p^k}\right\rfloor\right) =: \Sigma_1 + \Sigma_2. \tag{2.2}$$

The contribution of the small primes can be bounded easily. We have

$$\begin{aligned}\Sigma_1 &\leq \sum_{J<p\leq H}\left(\frac{h}{p^k}+1\right) \leq h\sum_{p>J}\frac{1}{p^k} + \pi(H) \\ &< h\left(\sigma_1 - \sum_{p\leq J}\frac{1}{p^k}\right) + \pi(H),\end{aligned} \tag{2.3}$$

where σ_1, the sum of the reciprocals of all of the primes to the k-th power, satisfies

$$\sigma_1 < \begin{cases} 0.4523 & \text{when } k=2, \\ 0.1748 & \text{when } k=3. \end{cases} \tag{2.4}$$

We group the sum over primes up to J appearing in (2.3) with $\sigma_0'(h,J)$ to write

$$\sigma_0(h,J) = 1 - \prod_{p\leq J}\left(1-\frac{1}{p^k}\right) - \sum_{p\leq J}\frac{1}{p^k} + \frac{2^{\pi(J)}}{h}, \tag{2.5}$$

so that we get

$$N_k(h,x) \le h\big(\sigma_0(h,J)+\sigma_1\big)+\pi(H)+\Sigma_2. \tag{2.6}$$

The term $\pi(H)$ above can be bounded with the help of the following well known result of Rosser and Schoenfeld [27, (3.2)].

Lemma 1. *For any $x>1$, one has*

$$\pi(x) < \frac{x}{\log x}\left(1+\frac{1.5}{\log x}\right). \tag{2.7}$$

Applying this lemma, we see that

$$\pi(H) < \sigma_2(h,m)h, \quad \sigma_2(h,m) := \frac{m}{\log(mh)}\left(1+\frac{1.5}{\log(mh)}\right). \tag{2.8}$$

The estimation of the sum Σ_2 occupies the remainder of the paper. We remark that primes $p > \sqrt[k]{2x}$ do not contribute to that sum, since for such primes we have

$$0 < \frac{x}{p^k} < \frac{x+h}{p^k} \le \frac{2x}{p^k} < 1.$$

Moreover, if $p > h^{1/k}$, we get

$$0 \le \left\lfloor \frac{x+h}{p^k} \right\rfloor - \left\lfloor \frac{x}{p^k} \right\rfloor \le \frac{h}{p^k}+1 < 2.$$

Thus, the finite sum Σ_2 counts the primes $p \in (H, \sqrt[k]{2x}]$ for which there exists an integer m with

$$\frac{x}{p^k} < m \le \frac{x+h}{p^k}.$$

The latter inequality can be expressed in terms of the fractional part of xp^{-k}: it says that $\{xp^{-k}\} > 1-hp^{-k}$. Therefore,

$$\Sigma_2 \le \big|S_k(H, \sqrt[k]{2x})\big|, \tag{2.9}$$

where

$$S_k(M,N) := \left\{ u \in \mathbb{Z} : M < u \le N,\ \gcd(u,2)=1,\ 1-\frac{h}{u^k} \le \left\{\frac{x}{u^k}\right\} < 1 \right\}. \tag{2.10}$$

We remark that while we no longer require the elements of $S_k(M,N)$ to be prime, we do restrict them to odd values so that the differences between any two elements of the set are even, a fact which will be useful later.

Thus, in view of (2.6), (2.8) and (2.9), to prove any of our results, it will suffice to find a choice of H such that

$$\big|S_k(H, \sqrt[k]{2x})\big| \le h\sigma_3(h,m), \tag{2.11}$$

for some bounded function $\sigma_3(h,m)$ such that

$$\sigma_0(h,J)+\sigma_1+\sigma_2(h,m)+\sigma_3(h,m) < 1-\frac{1}{h}. \tag{2.12}$$

In Sections 6–7, we establish inequalities of the form (2.11) and optimize the choices of several parameters to ensure that the respective versions of (2.12) hold. We conclude the present section with the statements of a couple of general-purpose lemmas, which we will use repeatedly in the remainder of the paper to obtain bounds on the spacing and cardinality of sets $S_k(M,N)$.

2.2. **Some general lemmas.** Our bounds on $|S_k(M,N)|$ are based on the simple idea that if the minimum distance between distinct elements of a set of integers A is at least d, then

$$|A \cap (M,N]| \le d^{-1}(N-M) + 1. \tag{2.13}$$

In Sections 3–5, we prove several results on the spacing between elements of sets

$$S_k(M) := S_k(M, \lambda M),$$

where $\lambda > 1$ is a constant. Those spacing estimates and inequality (2.13) yield bounds on $|S_k(M)|$, which we leverage with the help of the next lemma.

Lemma 2. *Suppose that A_1, A_2, A_3, b_1, b_2 are positive reals and u, v, λ are real numbers with $0 < u < v < 1 < \lambda$. Assume that for all $M \in [x^u, x^v]$ the estimate*

$$|S_k(M)| \le A_1 M^{b_1} + A_2 M^{-b_2} + A_3$$

holds. Then

$$|S_k(x^u, x^v)| \le A_1' x^{b_1 v} + A_2' x^{-b_2 u} + A_3' \log x + A_3,$$

where

$$A_1' = \frac{A_1}{1-\lambda^{-b_1}}, \quad A_2' = \frac{A_2}{1-\lambda^{-b_2}}, \quad A_3' = A_3 \cdot \frac{v-u}{\log \lambda}.$$

Proof. This is standard: we cover the interval $(x^u, x^v]$ with intervals of the form $(M, \lambda M]$, apply the hypothesis to each of them, and sum the ensuing geometric progressions. The only (minimal) novelty in the present version is the explicit description of the coefficients A_j' in terms of the A_j's and the various parameters. The reader will find a detailed proof of a variant for $\lambda = 2$ in [5, Lemma 1]. □

Some of our results also rely on the properties of divided differences. For a function $f : [a,b] \to \mathbb{R}$ and $s+1$ points $t_0, t_1, \dots, t_s \in [a,b]$, the *divided difference (of order s)*, $f[t_0, t_1, \dots, t_s]$, of f at the given points is defined recursively: we set $f[t_0] = f(t_0)$ when $s = 0$, and

$$f[t_0, t_1, \dots, t_s] = \frac{f[t_1, \dots, t_s] - f[t_0, \dots, t_{s-1}]}{t_s - t_0}$$

when $s \ge 1$. Divided differences are a tool in numerical analysis that has a long and rich history, but here we are interested only in two of their elementary properties, which we summarize in the next lemma. The reader can find proofs of these properties in many texts on numerical analysis that discuss interpolation theory: e.g., [17, Ch. 6].

Lemma 3. *Let $f : [a,b] \to \mathbb{R}$ be a function, let $t_0 < t_1 < \cdots < t_s$ be distinct numbers in $[a,b]$, and let $f[t_0,t_1,\ldots,t_s]$ denote the respective divided difference of f. Then*

$$f[t_0,t_1,\ldots,t_s] = \sum_{j=0}^{s} \frac{f(t_j)}{\prod_{i\neq j}(t_j - t_i)},$$

where the product is over $i \in \{0,1,\ldots,s\} \setminus \{j\}$. Moreover, if f has s continuous derivatives on $[a,b]$, then there is a number $\xi \in (t_0,t_s)$ such that

$$f[t_0,t_1,\ldots,t_s] = \frac{f^{(s)}(\xi)}{s!}.$$

3. Basic Spacing Lemmas

Let M be a large parameter, with $H \leq M \leq \sqrt[k]{2x}$, and let $\lambda \in (1,2]$ be a constant. In this section, we prove several lower bounds on the minimum distance between distinct elements of $S_k(M)$. As we pointed out in the introduction, the computational work in [23] allows us to assume that x is large. Also, while in our proofs we will utilize several different choices for h and H, we will always have $h \leq H$ and $h \leq 2x^{1/3}$. Thus, we assume in the remainder of the paper that

$$x \geq e^{41}, \quad 1000 \leq h \leq 2x^{1/3}. \tag{3.1}$$

3.1. **Spacing for pairs.** First, we show that two distinct elements of $S_k(M)$ cannot be "too close" to one another.

Lemma 4. *Let $k = 2$ or 3, and suppose that $H \leq M$. If u and $u + a$ are distinct elements of $S_k(M)$, then*

$$a > 0.999(kx)^{-1}M^{k+1}. \tag{3.2}$$

Proof. Consider the function $f(u) = xu^{-k}$. If $u, u+a \in S_k(M)$, we have

$$f(u) = n_1 - \theta_1, \quad f(u+a) = n_2 - \theta_2, \tag{3.3}$$

with $n_1, n_2 \in \mathbb{Z}$, $0 < \theta_1, \theta_2 < hM^{-k}$. So,

$$f(u+a) - f(u) = n - \theta, \qquad |\theta| < hM^{-k}.$$

By the mean-value theorem, there exists a number $\xi \in (u, u+a)$ such that

$$|f(u+a) - f(u)| = a|f'(\xi)| = \frac{kax}{\xi^{k+1}} > \frac{kx}{(\lambda M)^{k+1}}.$$

If $n = 0$, we have $|f(u+a) - f(u)| = |\theta| < hM^{-k}$, and we deduce that

$$k\lambda^{-k-1}x < hM < 3x^{5/6},$$

which contradicts (3.1). Thus, we have $n \neq 0$, so $|n| \geq 1$. We also get that

$$|\theta| < hM^{-k} \leq hH^{-k} \leq H^{1-k} \leq 0.001.$$

Hence, $|f(u+a)-f(u)| \geq 1-|\theta| \geq 0.999$, and we obtain

$$0.999 \leq |f(u+a)-f(u)| = kax\xi^{-k-1} < kaxM^{-k-1},$$

from which (3.2) follows. □

Applying (2.13) to the result of the last lemma, we obtain the following bound on the size of $S_k(M)$.

Corollary 1. *Under the hypotheses of Lemma 4, we have*

$$|S_k(M)| \leq c_{1,k}^{-1}(\lambda-1)xM^{-k}+1,$$

where $c_{1,2} = 0.4995$ *and* $c_{1,3} = 0.333$.

When $k = 3$, we can use the above lemma to prove the following alternative bound, which is stronger than the case $k = 3$ of (3.2) for $M \leq 1.5x^{2/7}$. This is the first of several results that make use of polynomial identities, similar to (3.5) below, which are inspired by the theory of Padé approximations. Such identities play a larger role in our proofs of gap results between cubefree integers than in the squarefree case. Moreover, they play a central role in our forthcoming work on gaps between k-free integers for general k.

Lemma 5. *Let* $\lambda \leq 1.2$, *and suppose that* $3\lambda^5 h \leq H \leq M$. *If* u *and* $u+a$ *are distinct elements of* $S_3(M)$, *then*

$$a > 0.7934x^{-1/3}M^{5/3}. \tag{3.4}$$

Proof. We recall the algebraic identity

$$\frac{a^3(2u+a)}{u^3(u+a)^3} = \frac{u+2a}{(u+a)^3} - \frac{u-a}{u^3}. \tag{3.5}$$

From this (and defining $n_1, n_2, \theta_1, \theta_2$ as in (3.3)), we deduce that

$$\begin{aligned}\frac{a^3(2u+a)x}{u^3(u+a)^3} &= f(u+a)(u+2a) - f(u)(u-a)\\ &= (n_2-\theta_2)(u+2a) - (n_1-\theta_1)(u-a) = n+\theta,\end{aligned} \tag{3.6}$$

where $n \in \mathbb{Z}$ and $\theta = (u-a)\theta_1 - (u+2a)\theta_2$. In particular, using that $0 < \theta_i < hM^{-3}$ and $u, u+a \in (M, 1.2M]$, we get

$$|\theta| \leq a(2\theta_1+\theta_2) + u|\theta_1-\theta_2| < (u+3a)hM^{-3} \leq 1.6hM^{-2}. \tag{3.7}$$

Next, we will prove that $n \neq 0$. We have

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} > \frac{2a^3ux}{u^3(u+a)^3} \geq \frac{2a^3x}{(\lambda M)^5} > 0.666\lambda^{-5}M^{-1}, \tag{3.8}$$

on recalling (3.2) and the trivial bound $a^2 \geq 1$. On the other hand, if $n = 0$, the right side of (3.8) equals θ (which must be positive), and (3.7) and (3.8) together yield

$$0.666\lambda^{-5} < 1.6hM^{-1} \leq 1.6hH^{-1},$$

which contradicts the hypothesis on H. Thus, we must have $|n| \geq 1$.

Similarly to (3.8), we find that

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} \leq \frac{2a^3(u+a)x}{u^3(u+a)^3} = \frac{2a^3x}{u^3(u+a)^2} \leq \frac{2a^3x}{M^5}. \tag{3.9}$$

On the other hand, the hypotheses of the lemma and (3.1) yield

$$|\theta| < 1.6hH^{-2} < 10^{-3},$$

so

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} \geq |n| - |\theta| > 0.999.$$

Combining the last inequality with (3.9), we deduce

$$a^3 > 0.4995x^{-1}M^5, \tag{3.10}$$

and the conclusion of the lemma follows. ☐

We can use this to obtain the following.

Corollary 2. *Under the hypotheses of Lemma 5, we have*

$$|S_3(M)| < 1.2604(\lambda-1)x^{1/3}M^{-2/3} + 1.$$

3.2. **Spacing for triples.** Next, we consider any three distinct elements $u, u+a, u+b$ of $S_k(M)$, with $0 < a < b$, and obtain lower bounds on b.

Lemma 6. *Let $\lambda \leq 1.2$, $m \geq 1.5$, and suppose that $mh = H \leq M$. If $0 < a < b$ and $u, u+a, u+b$ are elements of $S_2(M)$, then*

$$b \geq 1.3860x^{-1/3}M^{4/3}. \tag{3.11}$$

Proof. Suppose first that $b \leq 0.004M$. Write $u_1 = u$, $u_2 = u+a$, and $u_3 = u+b$, and let $n_1, n_2, n_3 \in \mathbb{Z}$ be such that

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-2} \qquad (i = 1, 2, 3).$$

We consider the second divided difference $f[u_1, u_2, u_3]$. By Lemma 3,

$$\begin{aligned} f[u_1,u_2,u_3] &= \frac{f(u_1)(u_3-u_2) + f(u_2)(u_1-u_3) + f(u_3)(u_2-u_1)}{(u_2-u_1)(u_3-u_1)(u_3-u_2)} \\ &= \frac{(n_1-\theta_1)(b-a) - (n_2-\theta_2)b + (n_3-\theta_3)a}{ab(b-a)} =: \frac{n-\theta}{V}, \end{aligned}$$

where

$$n = (b-a)n_1 - bn_2 + an_3 \qquad \text{and} \qquad \theta = (b-a)\theta_1 - b\theta_2 + a\theta_3.$$

In particular, since $\theta_i > 0$, we have

$$-bhM^{-2} < -b\theta_2 < \theta < (b-a)\theta_1 + a\theta_3 < bhM^{-2}.$$

Moreover, since u, $u+a$ and $u+b$ are all odd (see (2.10)) we know that a and b are both even, so n must be as well.

We will show that $n \neq 0$. Suppose that $n = 0$. Then

$$|f[u_1, u_2, u_3]| = \frac{|\theta|}{V} < \frac{bhM^{-2}}{ab(b-a)} = \frac{h}{a(b-a)M^2} < \frac{hx}{0.999M^5},$$

after an appeal to (3.2) and the bound $b - a \geq 2$. However, using Lemma 3, we also get that

$$|f[u_1, u_2, u_3]| = \frac{|f''(\xi)|}{2!} = \frac{3x}{\xi^4} \geq \frac{3x}{(\lambda M)^4}.$$

Thus,

$$\frac{3x}{(\lambda M)^4} < \frac{hx}{0.999M^5} < \frac{1.002hx}{HM^4},$$

which contradicts the hypotheses of the lemma.

Having proved that $n \neq 0$ and using that it is even, we find that $|n| \geq 2$. Hence,

$$|f[u_1, u_2, u_3]| = \frac{|n - \theta|}{V} \geq \frac{2 - |\theta|}{ab(b-a)} > \frac{1.997}{ab(b-a)}, \tag{3.12}$$

since

$$|\theta| < bhM^{-2} < 0.004hH^{-1} \leq \frac{1}{250m} \leq \frac{1}{375}.$$

On the other hand, by Lemma 3,

$$|f[u_1, u_2, u_3]| = \frac{3x}{\xi^4} \leq \frac{3x}{M^4}. \tag{3.13}$$

From (3.12), (3.13), and the elementary inequality $a(b-a) \leq \frac{1}{4}b^2$, we deduce that

$$\frac{3b^3x}{4} \geq 3ab(b-a)x > 1.997M^4, \tag{3.14}$$

and the conclusion of the lemma follows in the case $b \leq 0.004M$.

Finally, when $b > 0.004M$, we have

$$b^3 > (0.004M)^3 > \frac{8}{3}x^{-1}M^4,$$

by the assumptions that $M \leq \sqrt{2x}$ and $x \geq e^{41}$. □

Note that the expression on the right side of (3.11) is a lower bound for the minimum distances between successive elements of the set $S_2'(M)$ containing every other element of $S_2(M)$. Since $|S_2(M)| \leq 2|S_2'(M)|$, this observation and (2.13) yields the following corollary.

Corollary 3. *Under the hypotheses of Lemma 6, we have*

$$|S_2(M)| \leq 1.4430(\lambda - 1)x^{1/3}M^{-1/3} + 2.$$

Lemma 7. *Let $\lambda \le 1.04$, $m \ge 3$, and suppose that $mh \le H \le M$. If $0 < a < b$ and $u, u+a, u+b$ are elements of $S_3(M)$, then*

$$ab^4 \ge c_2(m)x^{-1}M^6, \tag{3.15}$$

where $c_2(m) = 0.4 - 0.05m^{-1}$.

Proof. We begin with the identity

$$\frac{a^5}{u^3(u+a)^3} = \frac{6u^2 - 3au + a^2}{u^3} - \frac{6u^2 + 15ua + 10a^2}{(u+a)^3}. \tag{3.16}$$

By substitution, this yields also the two companion identities

$$\frac{b^5}{u^3(u+b)^3} = \frac{6u^2 - 3bu + b^2}{u^3} - \frac{6u^2 + 15ub + 10b^2}{(u+b)^3}; \tag{3.17}$$

$$\begin{aligned}\frac{(b-a)^5}{(u+a)^3(u+b)^3} &= \frac{6(u+a)^2 - 3(b-a)(u+a) + (b-a)^2}{(u+a)^3} \\ &\quad - \frac{6(u+a)^2 + 15(u+a)(b-a) + 10(b-a)^2}{(u+b)^3}.\end{aligned} \tag{3.18}$$

In order to cancel out the higher order terms of u, we subtract (3.16) and (3.18) from (3.17). This gives

$$\begin{aligned}&\frac{(b-a)(a+b-3u)}{u^3} + \frac{b(5a-b+3u)}{(u+a)^3} + \frac{a(a-5b-3u)}{(u+b)^3} \\ &\quad = \frac{b^5}{u^3(u+b)^3} - \frac{a^5}{u^3(u+a)^3} - \frac{(b-a)^5}{(u+a)^3(u+b)^3}.\end{aligned} \tag{3.19}$$

For $u_1 = u$, $u_2 = u + a$, and $u_3 = u + b$, let

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-3} \qquad (i = 1, 2, 3).$$

Multiplying both sides of (3.19) by x, we have that

$$(b-a)(a+b-3u)f(u_1) + b(5a-b+3u)f(u_2) + a(a-5b-3u)f(u_3) =: n - \theta,$$

where

$$n = (b-a)(a+b-3u)n_1 + b(5a-b+3u)n_2 + a(a-5b-3u)n_3,$$

(note that n must be even, as a and b are even) and

$$\begin{aligned}|\theta| &= |b(3u-b)(\theta_2 - \theta_1) + 5ab(\theta_2 - \theta_3) + a(3u-a)(\theta_1 - \theta_3)| \\ &< hM^{-3}(3bu + 3au + 3ab - (a-b)^2) < 3bhM^{-3}(2u + a) \\ &< 6bhM^{-3}(u+b) < 6\lambda bhM^{-2}.\end{aligned} \tag{3.20}$$

Next, we show that $n \neq 0$. Suppose not. A direct check reveals that

$$\frac{b^5}{u^3(u+b)^3} - \frac{a^5}{u^3(u+a)^3} - \frac{(b-a)^5}{(u+a)^3(u+b)^3} = \frac{P(u;a,b)}{u^3(u+a)^3(u+b)^3},$$

where

$$\begin{aligned} P(u) &= 5ab(b-a)(b^2-ab+a^2)u^3 + 3ab(b^4-a^4)u^2 \\ &\quad + 3a^2b^2(b^3-a^3)u + a^3b^3(b^2-a^2) \\ &\quad > 5ab(b-a)(b^2-ab+a^2)u^3 \geq 5a^2b^2(b-a)u^3 > 10a^3bu^3. \end{aligned} \tag{3.21}$$

In particular, the two sides of (3.19) are positive. When $n = 0$, this entails that $\theta < 0$, and

$$-\theta = \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} > \frac{10a^3bu^3x}{u^3(u+a)^3(u+b)^3} \geq \frac{10b(0.4995M^5)}{(\lambda M)^6},$$

after appeals to (3.21) and (3.10). Hence, comparing the bound above to (3.20),

$$4.995\lambda^{-6}bM^{-1} < 6\lambda bhM^{-2} \leq 6\lambda bh(HM)^{-1},$$

or

$$m = \frac{H}{h} < \frac{6\lambda^5}{4.995}$$

which contradicts the hypotheses of the lemma. As such, we must have that $|n| \geq 2$.

On the other hand, bounding the numerator $P(u;a,b)$ from above, we find that

$$\begin{aligned} P(u) &\leq 5ab(b-a)(b^2-ab+a^2)u^3 + 3ab^5u^2 + 3a^2b^5u + a^3b^5 \\ &\leq 5ab^3(b-a)u^3 + 3ab^5u^2 + 3ab^6u + ab^7 \\ &< ab^4(5u^3 + 3bu^2 + 3b^2u + b^3) < 5ab^4(u+b)^3. \end{aligned}$$

Thus,

$$n - \theta = \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} \leq \frac{5ab^4x}{u^3(u+a)^3} < 5ab^4xM^{-6}. \tag{3.22}$$

Further, recalling that $b < (\lambda-1)M$, we deduce that

$$|\theta| \leq 6\lambda bhM^{-2} < 0.25hM^{-1} \leq 0.25hH^{-1}. \tag{3.23}$$

From (3.22) and (3.23), we deduce that

$$5ab^4xM^{-6} \geq \frac{P(u;a,b)x}{u^3(u+a)^3(u+b)^3} \geq 2 - |\theta| > 2 - 0.25hH^{-1},$$

which implies the conclusion of the lemma. □

Corollary 4. *Under the hypotheses of Lemma 7, we have*

$$|S_3(M)| \leq 2(\lambda-1)c_2(m)^{-1/5}x^{1/5}M^{-1/5} + 2.$$

4. Spacing for Pairs of Pairs

In this section, we study a special family of quadruples u, $u+a$, $u+b$, $u+a+b$ of elements of $S_k(M)$. The special form of the spacing between the four numbers allows us to obtain bounds on b that are stronger than those for general quadruples in $S_k(M)$; in the next section, we will average these bounds over b.

4.1. **Pairs in $S_2(M)$.** In the next lemma, we use the third-order divided difference of $f(u) = xu^{-2}$ for the points u, $u+a$, $u+b$, and $u+a+b$ to bound b from below.

Lemma 8. *Let $\lambda \leq 1.05$, $m \geq 5$, and suppose that $mh \leq H \leq M$. If $0 < a < 2a \leq b$ and $u, u+a, u+b, u+a+b$ are elements of $S_2(M)$, then*

$$ab^3 \geq 0.6600x^{-1}M^5. \tag{4.1}$$

Proof. Consider points $u_1 = u$, $u_2 = u+a$, $u_3 = u+b$, and $u_4 = u+a+b$ in $S_2(M)$. Recall that by the definition of the set $S_2(M)$, there exist integers $n_1, \dots, n_4$ and reals $\theta_1, \dots, \theta_4$ such that

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-2} \qquad (1 \leq i \leq 4). \tag{4.2}$$

We consider the divided difference $f[u_1, \dots, u_4]$.

Due to the special configuration of the distances between the four points, the formula in Lemma 3 simplifies to

$$f[u_1, u_2, u_3, u_4] = \frac{f(u_4) - f(u_1)}{ab(a+b)} - \frac{f(u_3) - f(u_2)}{ab(b-a)} =: \frac{n - \theta}{V},$$

where $V = ab(a+b)(b-a)$ and

$$n = (b-a)(n_4 - n_1) - (a+b)(n_3 - n_2),$$
$$\theta = (b-a)(\theta_4 - \theta_1) - (a+b)(\theta_3 - \theta_2).$$

We remark that n is an even integer and $|\theta| < 2bhM^{-2}$.

We will show that $n \neq 0$. Suppose that $n = 0$. Then

$$|f[u_1, \dots, u_4]| = \frac{|-\theta|}{V} \leq \frac{2bhM^{-2}}{ab(a+b)(b-a)}.$$

Recalling (3.14), we deduce that

$$|f[u_1, \dots, u_4]| < \frac{2hM^{-2}}{ab(b-a)} \leq \frac{6hx}{1.997M^6}.$$

However, Lemma 3 gives

$$|f[u_1, \dots, u_4]| = \frac{|f^{(3)}(\xi)|}{3!} = \frac{4x}{\xi^5} \geq \frac{4x}{(\lambda M)^5},$$

for some $\xi \in (M, \lambda M]$. We combine these upper and lower bounds to get

$$\frac{4x}{(\lambda M)^5} < \frac{6hx}{1.997M^6} < \frac{3.005hx}{HM^5},$$

which contradicts the assumptions of the lemma.

Since n is even and nonzero we can now use that $|n| \geq 2$ combined with the observation $b^2 - a^2 \geq 0.75b^2$ to obtain

$$|f[u_1, \dots, u_4]| = \frac{|n - \theta|}{V} \geq \frac{2 - |\theta|}{ab(b^2 - a^2)} \geq \frac{1.98}{0.75ab^3}, \tag{4.3}$$

since

$$|\theta| < 2bhM^{-2} < 2(\lambda-1)hH^{-1} \le \frac{1}{10m} \le \frac{1}{50}.$$

On the other hand, by Lemma 3,

$$|f[u_1,\dots,u_4]| = \frac{4x}{\xi^5} \le \frac{4x}{M^5}. \tag{4.4}$$

The lemma follows from (4.3) and (4.4). □

Our next result is of a somewhat different nature from the spacing lemmas established hitherto. In this lemma, instead of proving that the distance b between the two pairs exceeds some lower bound in terms of x, M, and possibly, a, we establish a kind of a dichotomy for b: either $b \ge B_1$ for some lower bound B_1, or $b \le B_2$, with B_2 significantly smaller than B_1.

Lemma 9. *Let $\lambda \le 1.05$, $m \ge 5$, and suppose that $mh \le H \le M$. If $0 < a < 2a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S_2(M)$, then exactly one of the conditions*

$$a^3 b < \lambda^6 h x^{-1} M^4, \tag{4.5}$$

or

$$a^3 b > (0.5 - \lambda m^{-1}) x^{-1} M^5, \tag{4.6}$$

must hold.

Proof. We start from the algebraic identity

$$\frac{2u+3a}{(u+a)^2} - \frac{2u-a}{u^2} = \frac{a^3}{u^2(u+a)^2}.$$

Since $u, u+a \in S_2(M)$, we can use this identity and (4.2) to get that

$$\frac{a^3 x}{u^2(u+a)^2} = \frac{(2u+3a)x}{(u+a)^2} - \frac{(2u-a)x}{u^2} = n' + \theta' \tag{4.7}$$

where $n' = (2u-3a)n_2 - (2u-a)n_1$ is an even integer and

$$|\theta'| = |\theta_1(2u-a) - \theta_2(2u+3a)| \le 2u|\theta_1 - \theta_2| + a(\theta_1 + 3\theta_2) < (2u+4a)hM^{-2}.$$

Combining (4.7) with the analogous identity for the pair $u+b, u+a+b$, we find that

$$\frac{a^3 x}{u^2(u+a)^2} - \frac{a^3 x}{(u+b)^2(u+a+b)^2} = n + \theta, \tag{4.8}$$

where $n \in \mathbb{Z}$ is even and

$$|\theta| < (4u+2b+8a)hM^{-2} \le 4(u+a+b)hM^{-2} \le 4\lambda hM^{-1} \le 4\lambda m^{-1}.$$

Next we observe, by the mean-value theorem, there is a $\xi \in (u, u+b)$ such that

$$\frac{a^3 x}{u^2(u+a)^2} - \frac{a^3 x}{(u+b)^2(u+a+b)^2} = \frac{2a^3 bx(2\xi+a)}{\xi^3(\xi+a)^3}.$$

This expression is bounded above by

$$\frac{2a^3bx(2\xi+a)}{\xi^3(\xi+a)^3} < \frac{4a^3bx}{\xi^3(\xi+a)^2} < 4a^3bxM^{-5}, \tag{4.9}$$

and bounded below by

$$\frac{2a^3bx(2\xi+a)}{\xi^3(\xi+a)^3} > \frac{4a^3bx}{\xi^2(\xi+a)^3} > 4a^3bx(\lambda M)^{-5}. \tag{4.10}$$

When $a^3b \le (0.5 - \lambda m^{-1})x^{-1}M^5$, (4.8), (4.9), and the bound on $|\theta|$ yield

$$n - 4\lambda m^{-1} < n + \theta < 4a^3bxM^{-5} \le 2 - 4\lambda m^{-1},$$

and hence, $n < 2$. On the other hand, if $a^3b \ge \lambda^6 hx^{-1}M^4$, we deduce from (4.8) and (4.10) that

$$4\lambda hM^{-1} < 4a^3bx(\lambda M)^{-5} < n + \theta < n + 4\lambda hM^{-1},$$

so in this case $n > 0$. Since n is an even integer, it can satisfy only one of the conditions $n > 0$ and $n < 2$; therefore, at least one of (4.5) or (4.6) must hold. This completes the proof, since under the hyptheses of the lemma, the lower bound in (4.6) exceeds the upper bound in (4.5) at least by a constant factor. □

4.2. **Pairs in $S_3(M)$.** We consider $u, u+a, u+b, u+a+b \in S_3(M)$, with $0 < a \le b$. We use several algebraic identities to obtain a series of lower bounds on products of the form a^ib^j.

Lemma 10. *Let $\lambda \le 1.04$, $m \ge 3$, and suppose that $mh \le H \le M$. If $0 < a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S_3(M)$, then*

$$a^3b > 0.0999x^{-1}M^6. \tag{4.21}$$

Proof. Note that under the hypotheses of the lemma, we have

$$a \le \frac{1}{2}(a+b) < 0.5(\lambda - 1)M. \tag{4.22}$$

Recall that, by (3.6) and (3.7), we have

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} = n_1 + \theta_1$$

with $n_1 \in \mathbb{Z}$ and $|\theta_1| < 1.6hM^{-2}$. Indeed, using (4.22) we can strengthen this to $|\theta_1| < 1.06hM^{-2}$. Similarly,

$$\frac{a^3(2u+2b+a)x}{(u+b)^3(u+a+b)^3} = n_2 + \theta_2$$

with $n_2 \in \mathbb{Z}$ and $|\theta_2| < 1.06hM^{-2}$. Combining these identities, we find that

$$\frac{a^3(2u+a)x}{u^3(u+a)^3} - \frac{a^3(2u+2b+a)x}{(u+b)^3(u+a+b)^3} = n + \theta$$

with $|\theta| \le |\theta_1| + |\theta_2| < 2.12hM^{-2}$. Now, we note that when the right side of the above identity is combined into a single fraction, the numerator can be written as

$$a^3x[4bu^3(u+a)^3 + 3abu^2(u+a)^3 + 3b(2u+a)u^3(u+a)^2 + \text{other positive terms}].$$

Since

$$3abu^2(u+a)^3 + 3b(2u+a)u^3(u+a)^2 > 6bu^3(u+a)^3,$$

we can use (3.10) to obtain the lower bound

$$\begin{aligned} n+\theta &\ge \frac{10a^3bu^3(u+a)^3x}{u^3(u+a)^3(u+b)^3(u+a+b)^3} \\ &= \frac{10a^3bx}{(u+b)^3(u+a+b)^3} \ge \frac{20a^3x}{(\lambda M)^6} \ge \frac{9.99}{\lambda^6 M}. \end{aligned}$$

In particular, if $n = 0$, we see that

$$9.99\lambda^{-6} \le M\theta < 2.12hM^{-1} < 2.12m^{-1},$$

which contradicts the hypotheses. Thus, we must have $|n| \ge 1$.

We now observe that (note the first equality can be checked using a computer algebra system)

$$\begin{aligned} &(2u+a)(u+b)^3(u+a+b)^3 - (2u+2b+a)u^3(u+a)^3 \\ &\quad = 10bu^3(u+a+b)^3 + b(3(a+b)^2 + 7b^2 + 9ab)u^2(u+a+b)^2 \\ &\qquad + b^2(3(a+b)^3 + ab^2 + 2a^2b - b^3)u(u+a+b) + ab^3(a+b)^3 \\ &\quad < 10bu^3(u+a+b)^3 + 10b(a+b)^2u^2(u+a+b)^2 \\ &\qquad + 3b^2(a+b)^3u(u+a+b) + ab^3(a+b)^3 \\ &\quad < 10bu^3(u+a+b)^3 + b^2u^2(u+a+b)^3 + b^3u(u+a+b)^3 + b^4(a+b)^3 \\ &\quad < b(u+a+b)^3(10u^3 + bu^2 + b^2u + b^3) < 10b(u+b)^3(u+a+b)^3. \end{aligned}$$

Thus,

$$1 - |\theta| \le n + \theta \le \frac{10a^3bx}{u^3(u+a)^3} \le 10a^3bxM^{-6}.$$

Since $|\theta| \le 2.12hH^{-2} < 0.001$, we conclude that

$$0.999 < 10a^3bxM^{-6},$$

and the lemma follows. □

Lemma 11. *Let $\lambda \le 1.04$, $m \ge 8$, and suppose that $mh \le H \le M$. If $0 < a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S_3(M)$, then*

$$a^3b^3 > 0.0664x^{-1}M^7. \tag{4.23}$$

Proof. First, we observe that if $b \geq 0.01M$, Lemma 10 gives

$$a^3b^3 \geq 0.0999b^2x^{-1}M^6 \geq 0.000009x^{-1}M^8 > 0.07x^{-1}M^7,$$

since $M \geq mh \geq 8000$. Thus, we may assume for the remainder of the proof that $0 < a \leq b < 0.01M$. Let $u_1 = u$, $u_2 = u + a$, $u_3 = u + b$, and $u_4 = u + a + b$, and recall that by the definition of the set $S_3(M)$, there exist integers $n_1, \dots, n_4$ and reals $\theta_1, \dots, \theta_4$ such that

$$f(u_i) = n_i - \theta_i, \quad 0 < \theta_i < hM^{-3} \qquad (1 \leq i \leq 4). \tag{4.24}$$

We begin by constructing a rational function of the form

$$R(u; a, b) = \frac{P_1(u; a, b)}{u^3} + \frac{P_2(u; a, b)}{(u + a)^3} + \frac{P_3(u; a, b)}{(u + b)^3} + \frac{P_4(u; a, b)}{(u + a + b)^3}, \tag{4.25}$$

where $P_i(u; a, b)$ are homogeneous quadratic polynomials, which are at most linear in u. Clearly, any such rational function can be rewritten as

$$R(u; a, b) = \frac{C(u; a, b)}{u^3(u + a)^3(u + b)^3(u + a + b)^3}, \tag{4.26}$$

for some homogeneous polynomial $C(u; a, b)$ of total degree 11, which has at most degree 10 in u. We choose the polynomials $P_i(u; a, b)$ as to minimize the degree of C with respect to u. The identities

$$\frac{u + 2a}{(u + a)^3} - \frac{u - a}{u^3} = \frac{a^3(2u + a)}{u^3(u + a)^3}, \quad \frac{1}{u^3} - \frac{1}{(u + a)^3} = \frac{a(3u^2 + 3ua + a^2)}{u^3(u + a)^3},$$

imply that any choice of the form

$$P_1(u, a, b) = \alpha(-u + a) - \beta, \qquad P_2(u, a, b) = \alpha(u + 2a) + \beta,$$
$$P_3(u, a, b) = P_1(u + b, a, b) + 2\beta, \qquad P_4(u, a, b) = P_2(u + b, a, b) - 2\beta,$$

where α, β depend only on a, b, reduces the u-degree of C to at most 7. The choice $\alpha = 3b$ and $\beta = a^2$ then ensures that the coefficients of u^7 and u^6 in $C(u; a, b)$ also cancel out. Thus, we choose

$$P_1(u, a, b) = -3bu + 3ab - a^2, \qquad P_2(u, a, b) = 3bu + 6ab + a^2,$$
$$P_3(u, a, b) = -3bu - 3b^2 + 3ab + a^2, \qquad P_4(u, a, b) = 3bu + 3b^2 + 6ab - a^2.$$

With the above choice, a straightforward (but tedious) direct calculation reveals that

$$C(u; a, b) = 6a^3b(5b^2 - a^2)v^5 - 15a^3b(a + b)(5b^2 - a^2)v^4 + D(v; a, b), \tag{4.27}$$

where $v = u + a + b$ and $D(v; a, b)$ is a homogeneous polynomial of total degree 11, which is cubic in v. In particular, the coefficient of v^3 in $D(v; a, b)$ is

$$72a^3b^5 + 150a^4b^4 + 52a^5b^3 - 30a^6b^2 - 12a^7b < 74a^3b^3(a + b)^2,$$

on recalling that $0 < a \leq b$. Note that we have also

$$72a^3b^5 + 150a^4b^4 + 52a^5b^3 - 30a^6b^2 - 12a^7b > 58a^3b^3(a + b)^2.$$

Similarly, the coefficient of v^2 is bounded above and below as

$$-36a^3b^3(a+b)^3 < 3a^8b+18a^7b^2-3a^6b^3-93a^5b^4-108a^4b^5-33a^3b^6 < -27a^3b^3(a+b)^3;$$

the coefficient of v is bounded above and below as

$$6a^3b^4(a+b)^3 < -3a^8b^2-6a^7b^3+18a^6b^4+48a^5b^5+33a^4b^6+6a^3b^7 < 14a^3b^4(a+b)^3;$$

and the constant (in v) terms are bounded as

$$-4.25a^4b^5(a+b)^2 < a^8b^3-6a^6b^5-8a^5b^6-3a^4b^7 < -3a^4b^5(a+b)^2.$$

Moreover, since $M < u < v \le \lambda M$ and $0 < a, b < 0.01M$, we find that

$$a^4b^5(a+b)^2 < 0.01a^3b^4(a+b)^3v, \quad a^3b^4(a+b)^3v < 0.01a^3b^3(a+b)^3v^2,$$
$$a^3b^3(a+b)^3v^2 < 0.02a^3b^3(a+b)^2v^3.$$

From these and the earlier bounds on the coefficients of $D(v; a, b)$, we deduce that

$$0 < D(v; a, b) < 74a^3b^3(a+b)^2v^3. \tag{4.28}$$

Inserting this upper bound into (4.27) we now have

$$\begin{aligned} C(u; a, b) &= 6a^3b(5b^2-a^2)v^5 - 15a^3b(a+b)(5b^2-a^2)v^4 + D(v; a, b) \\ &\le a^3bv^3\big(6(5b^2-a^2)v^2 - 15(a+b)(5b^2-a^2)v + 74b^2(a+b)^2\big). \end{aligned}$$

We expand $v = u + a + b$ (and use $a + b < 0.02u$) to bound the term in parentheses above as

$$\begin{aligned} &6(5b^2-a^2)v^2 - 15(a+b)(5b^2-a^2)v + 74b^2(a+b)^2 \\ &\quad = 6(5b^2-a^2)u^2 - 3(a+b)(5b^2-a^2)u - 9(a+b)^2(5b^2-a^2) + 74b^2(a+b)^2 \\ &\quad \le 6(5b^2-a^2)u^2 - 3(a+b)(5b^2-a^2)u + 38b^2(a+b)^2 \\ &\quad < 6(5b^2-a^2)u^2 - 12(a+b)b^2u + (a+b)b^2u < 30b^2u^2. \end{aligned}$$

Thus $C(u; a, b) < 30a^3b^3u^2v^3$. We can also use (4.28) to bound C from below:

$$\begin{aligned} C(u; a, b) &= 3a^3b(5b^2-a^2)v^4(2v - 5(a+b)) + D(v; a, b) \\ &> 12a^3b^3v^4(2u - 3(a+b)) \\ &> 12a^3b^3v^2\big(u^2 + 2(a+b)u\big)(2u - 3(a+b)) \\ &> 12a^3b^3uv^2\big(2u^2 + 0.98(a+b)u\big) > 24a^3b^3u^3v^2. \end{aligned}$$

From these bounds and (4.26), we deduce that

$$18.238a^3b^3M^{-7} < \frac{24a^3b^3}{(\lambda M)^7} \le R(u; a, b) \le 30a^3b^3M^{-7}. \tag{4.29}$$

On the other hand, multiplying both sides of (4.25) by x, we get from (4.24) that

$$xR(u; a, b) = n - \theta$$

where $n \in \mathbb{Z}$ is even (since all of the coefficients in each of the P_i are even) and

$$\theta = \sum_{i=1}^{4} P_i(u; a, b)\theta_i.$$

We can bound $|\theta|$ as follows:

$$\begin{aligned}|\theta| &\le 3bu|\theta_1 - \theta_2| + 3b(u+b)|\theta_3 - \theta_4| \\ &\quad + 3ab(\theta_1 + 2\theta_2 + \theta_3 + 2\theta_4) + a^2|\theta_1 - \theta_2 - \theta_3 + \theta_4| \\ &< \big(3b(u+b) + 3bu + 18ab + 2a^2\big)hM^{-3} \\ &< (6b(u+a+b) + 11ab)hM^{-3} < (6\lambda + 0.11)bhM^{-2},\end{aligned}$$

upon recalling that $u + a + b < \lambda M$ and $a < 0.01M$.

Next, we will show that $n \neq 0$. Suppose that $n = 0$. Then we have

$$18.238a^3b^3xM^{-7} \le |Cx| = |\theta| \le (6\lambda + 0.11)bhM^{-2} \le 6.35bhM^{-2}.$$

Combining this inequality with (4.21), we obtain

$$1.822b^2M^{-1} < 18.238a^3b^3xM^{-7} \le 6.35b(mM)^{-1},$$

which is a contradiction under the hypotheses of the lemma. Thus $n \ge 2$, and we get

$$Cx = n - \theta > 2 - 6.35bhM^{-2} > 2 - 0.0635m^{-1} > 1.992.$$

Combining this with the upper bound in (4.29), we get that

$$1.992 < Cx < 30a^3b^3xM^{-7},$$

and the desired conclusion follows. □

Next, we prove a version of Lemma 9. Note that in this case, the conditions (4.30) and (4.31) are not always mutually exclusive—when M is close to H and $m \le 15$, the two inequalities may hold simultaneously.

Lemma 12. *Let $\lambda \le 1.04$, $m \ge 8$, and suppose that $mh \le H \le M$. If $0 < a \le b$ and $u, u+a, u+b, u+a+b$ are elements of $S_3(M)$, then at least one of the conditions*

$$a^5b < 2\lambda^9hx^{-1}M^6, \tag{4.30}$$

or

$$a^5b > (\tfrac{1}{3} - 2\lambda^2m^{-1})x^{-1}M^7, \tag{4.31}$$

must hold.

Proof. As in the proof of the last lemma, let $u_1 = u$, $u_2 = u + a$, $u_3 = u + b$, and $u_4 = u + a + b$, and recall (4.24). We rely on the identity

$$\begin{aligned}\frac{a^5x}{u^3(u+a)^3} &= \frac{(6u^2 - 3au + a^2)x}{u^3} - \frac{(6u^2 + 15ua + 10a^2)x}{(u+a)^3} \\ &= (6u^2 - 3ua + a^2)f(u) - (6u^2 + 15ua + 10a^2)f(u+a) = n' - \theta',\end{aligned}$$

with $n' \in \mathbb{Z}$, even and

$$\theta' = (6u^2 - 3ua + a^2)\theta_1 - (6u^2 + 15ua + 10a^2)\theta_2.$$

Using this relation and the analogous one for the pairs $u+b, u+a+b$, we find that

$$\frac{xa^5}{u^3(u+a)^3} - \frac{xa^5}{(u+b)^3(u+a+b)^3} = n + \theta, \tag{4.32}$$

with $n \in \mathbb{Z}$ even and

$$\begin{aligned} |\theta| &< (6u^2 + 6(u+b)^2 + 12au + 15ab + 11a^2)hM^{-3} \\ &< (6(u+a+b)^2 + 6(u+a)^2)hM^{-3} < 12\lambda^2 hM^{-1}. \end{aligned}$$

By the mean-value theorem,

$$(u+b)^3(u+a+b)^3 - u^3(u+a)^3 = 3b\xi^2(\xi+a)^2(2\xi+a)$$

for some $\xi \in (u, u+b)$, so

$$6bu^3(u+a)^2 < (u+b)^3(u+a+b)^3 - u^3(u+a)^3 < 6b(u+b)^2(u+a+b)^3.$$

We use this to bound the left side of (4.32) and get that

$$n + \theta \le \frac{6a^5bx}{u^3(u+a)^3(u+b)} < 6a^5bxM^{-7}.$$

Similarly, we have

$$n + \theta \ge \frac{6a^5bx}{(u+a)(u+b)^3(u+a+b)^3} > 6\lambda^{-7}a^5bxM^{-7}.$$

Since $M \ge H$, we have that $|\theta| < 12\lambda^2 m^{-1}$. So, it follows from the upper bound on $n+\theta$ that if $b \le (\frac{1}{3} - 2\lambda^2 m^{-1})a^{-5}x^{-1}M^7$, we have

$$n + \theta < 6a^5bxM^{-7} < 2 - |\theta|;$$

hence, $n < 2$. On the other hand, if $b \ge 2\lambda^9 a^{-5}hx^{-1}M^6$, then using the lower bound for $n+\theta$, we have that

$$n + \theta > 6\lambda^{-7}a^5bxM^{-7} \ge 12\lambda^2 hH^{-1} > |\theta|,$$

implying that $n > 0$. Since $n > 0$ and $n < 2$ cannot occur simultaneously when n is even, the lemma follows. □

5. The Main Bounds on $|S_k(M)|$

Let

$$A_2 = 1.3860x^{-1/3}M^{4/3}, \quad A_3 = (0.4 - 0.05m^{-1})^{1/5}x^{-1/5}M^{6/5}. \tag{5.1}$$

In Section 3, we proved that $b \ge A_k$ whenever $u, u+a, u+b$ are distinct elements of $S_k(M)$. Therefore, if $u_0, u_1, \dots, u_s$ are the elements of $S_k(M)$, listed in increasing order, the set $S'_k(M) = \{u_0, u_2, u_4, \dots\}$ has no gaps $< A_k$ and satisfies

$$|S_k(M)| \le 2|S'_k(M)|. \tag{5.2}$$

In this section, we use (5.2) and the lemmas in the last section to prove the following results.

Proposition 1. *Suppose* $h = 11x^{1/5}\log x$*, let* $\lambda = 1.045$ *and* $x \geq e^{116}$*, and suppose that* $5.5h \leq M \leq x^{2/5}$*. Then*

$$S_2(M) \leq h(\sigma_3(M) + \sigma_4(M)), \tag{5.3}$$

where

$$\sigma_3(M) = \big(0.5298x^{1/5} + 0.3400x^{-1/5}M\big)h^{-1} + 0.0308x^{1/15}M^{-1/3}, \tag{5.4}$$

and

$$\sigma_4(M) = \begin{cases} 1.2105x^{-2/3}M^{7/3} & \text{if } M \leq 5x^{1/4}, \\ 1.4182x^{-1/9}M^{1/9} & \text{if } M > 5x^{1/4}. \end{cases} \tag{5.5}$$

Proposition 2. *Suppose* $h = 5x^{1/7}\log x$*, let* $\lambda = 1.04$ *and* $x \geq e^{200}$*, and suppose that* $11h \leq M \leq x^{2/7}$*. Then*

$$S_3(M) \leq h(\sigma_3(M) + \sigma_4(M)), \tag{5.6}$$

where

$$\sigma_3(M) = \big(0.4212x^{1/7} + 0.1900x^{-1/7}M\big)h^{-1} + 0.0374x^{1/21}M^{-1/3}, \tag{5.7}$$

and

$$\sigma_4(M) = \begin{cases} 0.9519x^{-2/3}M^{11/3} & \text{if } M \leq 18x^{1/6}, \\ 5.1698x^{-1/15}M^{1/15} & \text{if } M > 18x^{1/6}. \end{cases} \tag{5.8}$$

Remark 1. Notice that when x is relatively small, the conditions $M \leq 5x^{1/4}$ (in Proposition 1) and $M \leq 18x^{1/6}$ (in Proposition 2) are impossible, and so only the second condition will be used in the range of "small" values of x.

The proofs of these propositions use the set

$$T_k(M; a) = \{u : u, u + a \text{ are consecutive elements of } S'_k(M)\}$$

to bound $|S'_k(M)|$. The starting point is the elementary identity

$$|S'_k(M)| = 1 + \sum_{a=1}^{\infty} |T_k(M; a)| = 1 + \sum_{a \geq A_k} |T_k(M; a)|, \tag{5.9}$$

which is a direct consequence of the definition of $T_k(M; a)$. Further, for any $B \geq A_k$, we have

$$\sum_{a \geq B} a|T_k(M; a)| \leq \sum_{a \geq A_k} a|T_k(M; a)| \leq (\lambda - 1)M + 1,$$

so

$$\sum_{a \geq B} |T_k(M; a)| \leq (\lambda - 1)MB^{-1} + B^{-1}.$$

Applying this inequality to the right side of (5.9), we find that, for any parameter $B \geq 2$,

$$|S_k'(M)| \leq 1.5 + (\lambda - 1)MB^{-1} + \sum_{A_k \leq a < B} |T_k(M; a)|. \tag{5.10}$$

5.1. **Proof of Proposition 1.** In this proof, we write A for the quantity A_2 defined in (5.1), and we select

$$B = \delta x^{-1/5} M, \quad \delta = 0.17, \tag{5.11}$$

in the imminent application of (5.10). We fix an integer a, with $A \leq a \leq B$. If $u_0, u_1, \dots, u_t$ are the elements of $T_2(M; a)$, listed in increasing order, the set $T'(M) = \{u_0, u_2, u_4, \dots\}$ contains only elements of $T_2(M; a)$ such that if $u, u + b \in T'(M)$, then $b \geq 2a$. Clearly, $|T_2(M; a)| \leq 2|T'(M)|$.

Let I be a subinterval of $(M, \lambda M]$ of length

$$|I| = (0.5 - \lambda m^{-1})a^{-3}x^{-1}M^5,$$

and let $u, u + b$ be two elements of $T'(M) \cap I$. Since $b \geq 2a$, we can apply Lemma 9 to show that b must satisfy (4.5). Taking u and $u + b$ to be the smallest and largest elements of $T'(M) \cap I$ respectively, we can use this bound on b to deduce that the set $T'(M) \cap I$ is contained in an interval of length $\leq \lambda^6 a^{-3} h x^{-1} M^4$. Furthermore, by (4.1), we have that

$$b \geq 0.8706 a^{-1/3} x^{-1/3} M^{5/3}.$$

Combining these two observations we find that

$$|T'(M) \cap I| \leq \frac{\lambda^6 a^{-3} h x^{-1} M^4}{0.8706 a^{-1/3} x^{-1/3} M^{5/3}} + 1 < 1.4959 a^{-8/3} h x^{-2/3} M^{7/3} + 1. \tag{5.12}$$

Since we need at most

$$\frac{(\lambda - 1)M}{(0.5 - \lambda m^{-1}) a^{-3} x^{-1} M^5} + 1 < 0.1452 a^3 x M^{-4} + 1 \tag{5.13}$$

intervals of length $|I|$ to cover $(M, \lambda M]$, we conclude that

$$\begin{aligned} |T'(M)| &\leq \left(0.1452 a^3 x M^{-4} + 1\right)\left(1.4959 a^{-8/3} h x^{-2/3} M^{7/3} + 1\right) \\ &< 0.2173 a^{1/3} h x^{1/3} M^{-5/3} + 0.1452 a^3 x M^{-4} + 1.4959 a^{-8/3} h x^{-2/3} M^{7/3} + 1. \end{aligned}$$

Thus,

$$\begin{aligned} |T_2(M; a)| < 0.4346 a^{1/3} h x^{1/3} M^{-5/3} &+ 0.2904 a^3 x M^{-4} \\ &+ 2.9918 a^{-8/3} h x^{-2/3} M^{7/3} + 2. \end{aligned} \tag{5.14}$$

Next, we use (5.14) to bound the right side of (5.10). With our choice of parameters, (5.10) gives

$$S_2'(M) \leq 1.5 + 0.045\delta^{-1} x^{1/5} + \sum_{A \leq a < B} |T_2(M; a)|. \tag{5.15}$$

Thus, we need to sum each of the four terms on the right side of (5.14) over $a \in [A, B)$. Recalling the inequality

$$\sum_{k \le K} k^s < \frac{(K+1)^{s+1}}{s+1} \qquad (s > 0),$$

and noting that $B = \delta M x^{-1/5} \ge 5.5 h \delta x^{-1/5} > 10.285 \log x > 1193$, we find that

$$\sum_{\substack{2 \le a \le B \\ a \text{ even}}} a^s < \frac{(B+2)^{s+1}}{2(s+1)} < \frac{(1.002B)^{s+1}}{2(s+1)}. \tag{5.16}$$

Hence,

$$\begin{aligned} 0.4346 h x^{1/3} M^{-5/3} \sum_{\substack{A \le a < B \\ a \text{ even}}} a^{1/3} &< \frac{0.4346 \cdot (1.002B)^{4/3}}{8/3} h x^{1/3} M^{-5/3} \\ &< 0.0154 h x^{1/15} M^{-1/3}, \end{aligned} \tag{5.17}$$

and

$$0.2904 x M^{-4} \sum_{\substack{A \le a < B \\ a \text{ even}}} a^3 < \frac{0.2904 \cdot (1.002B)^4}{8} x M^{-4} < 0.00004 x^{1/5}. \tag{5.18}$$

Combining (5.11), (5.14), (5.15), (5.17), and (5.18), we conclude that

$$S_2'(M) \le h \sigma_3'(M) + 2.9918 h x^{-2/3} M^{7/3} \sum_{\substack{A \le a < B \\ a \text{ even}}} a^{-8/3}, \tag{5.19}$$

where

$$\sigma_3'(M) = \big(0.2649 x^{1/5} + 0.17 x^{-1/5} M\big) h^{-1} + 0.0154 x^{1/15} M^{-1/3}. \tag{5.20}$$

We estimate the sum on the right side of (5.19) in different ways, depending on the size of M. When $M \le 5x^{1/4}$, we use that

$$\sum_{\substack{A \le a < B \\ a \text{ even}}} a^{-8/3} < \frac{\zeta(8/3)}{2^{8/3}} < 0.2023. \tag{5.21}$$

On the other hand, when $M > 5x^{1/4}$, we have $A > 1.386 \cdot 5^{4/3} > 11.8501$, so

$$\sum_{\substack{A \le a < B \\ a \text{ even}}} a^{-8/3} < \frac{3}{5 \cdot 2^{8/3}} \left(\frac{A}{2} - 1 \right)^{-5/3} < 0.4083 A^{-5/3} < 0.237 x^{5/9} M^{-20/9}. \tag{5.22}$$

The proposition follows from (5.2) and (5.19)–(5.22). □

5.2. **Proof of Proposition 2.** Similarly to the proof of Proposition 1, we write A for the quantity A_3 defined in (5.1), and we select

$$B = \delta x^{-1/7} M, \quad \delta = 0.19, \tag{5.23}$$

in the application of (5.10). We now fix an integer a, with $A \le a \le B$. By Lemma 12, if we consider an interval I of length

$$|I| \le (\tfrac{1}{3} - 2\lambda^2 m^{-1}) x^{-1} M^7,$$

we must have $b < 2\lambda^9 h a^{-5} x^{-1} M^6$ for any elements $u, u+b \in T_3(M;a) \cap I$. Then we can use (4.23) to get

$$\begin{aligned} |T_3(M;a) \cap I| &\le \frac{2\lambda^9 h a^{-5} x^{-1} M^6}{(0.0664)^{1/3} a^{-1} x^{-1/3} M^{7/3}} + 1 \\ &< 7.0298 a^{-4} h x^{-2/3} M^{11/3} + 1. \end{aligned}$$

Since we need at most

$$\frac{(\lambda-1)M}{(\frac{1}{3} - 2\lambda^2 m^{-1}) x^{-1} M^7 a^{-5} x^{-1} M^7} + 1 < 0.2929 a^5 x M^{-6} + 1$$

intervals of length $|I|$ to cover $(M, \lambda M]$, we conclude that

$$\begin{aligned} |T_3(M;a)| &\le (0.2929 a^5 x M^{-6} + 1)(7.0298 a^{-4} h x^{-2/3} M^{11/3} + 1) \\ &\le 2.0591 a h x^{1/3} M^{-7/3} + 0.2929 a^5 x M^{-6} + 7.0298 a^{-4} h x^{-2/3} M^{11/3} + 1. \end{aligned} \tag{5.24}$$

Next, we use this to bound the right side of (5.10). With our choice of parameters, (5.10) yields

$$S_3'(M) \le 1.5 + 0.04\delta^{-1} x^{1/7} + \sum_{\substack{A \le a < B \\ a \text{ even}}} |T_3(M;a)|. \tag{5.25}$$

To sum each of the four terms on the right side of (5.24) over $a \in [A, B)$, we recall (5.16) and note that the constant 1.002 in that inequality can be reduced to 1.001 in the present context, since $B \ge 11\delta h x^{-1/7} > 2090$. We find that

$$\begin{aligned} 2.0591 h x^{1/3} M^{-7/3} \sum_{\substack{A \le a < B \\ a \text{ even}}} a &< \frac{2.0591(1.001B)^2}{4} x^{1/3} M^{-7/3} \\ &< 0.0187 h x^{1/21} M^{-1/3}, \end{aligned} \tag{5.26}$$

and

$$0.2929 x M^{-6} \sum_{\substack{A \le a < B \\ a \text{ even}}} a^5 < \frac{0.2929 \cdot (1.001B)^6}{12} x M^{-6} < 0.00001 x^{1/7}. \tag{5.27}$$

Combining (5.23)–(5.27), we conclude that

$$S_3'(M) \le h\sigma_3'(M) + 7.0298hx^{-2/3}M^{11/3} \sum_{\substack{A\le a<B \\ a \text{ even}}} a^{-4}, \tag{5.28}$$

where

$$\sigma_3'(M) = \left(0.2106x^{1/7} + 0.095x^{-1/7}M\right)h^{-1} + 0.0187x^{1/21}M^{-1/3}. \tag{5.29}$$

We estimate the sum on the right side of (5.28) in different ways, depending on the size of M. When $M \le 18x^{1/6}$, we use that

$$\sum_{\substack{A\le a<B \\ a \text{ even}}} a^{-4} < \frac{\zeta(4)}{16} = \frac{\pi^4}{1440} < 0.0677. \tag{5.30}$$

On the other hand, when $M > 18x^{1/6}$, we have $A > 0.8306 \cdot 18^{6/5} > 26.6513$, so

$$\sum_{\substack{A\le a<B \\ a \text{ even}}} a^{-4} < \frac{1}{3\cdot 2^4}\left(\frac{A}{2}-1\right)^{-3} < 0.2107A^{-3} < 0.3677x^{3/5}M^{-18/5}. \tag{5.31}$$

The proposition follows from (5.2) and (5.28)–(5.31). □

6. Proof of Theorem 1

The proof of the theorem uses different approaches for different values of x. As we stated in the introduction, the work in [23] establishes our result (and much more) for $x \le e^{41}$. In Section 6.1, we focus on large x and show that for $x \ge e^{116}$, Theorem 1 follows from Proposition 1. To complete the proof, in Section 6.2, we prove two asymptotically weaker variants, which are, however, stronger than the theorem for small x. Those alternative results establish Theorem 1 in the intermediate range $e^{41} \le x \le e^{116}$.

6.1. **Large x.** Let $x \ge e^{116}$ and set $H = 5.5h$ in (2.2) and (2.11). First, we use Proposition 1 and Lemma 2 to bound $S_2(H, x^{2/5})$.

Suppose first that $H \le 5x^{1/4}$, (in this case we can assume $x \ge e^{150}$) we split $S_2(H, x^{2/5})$ in two pieces to account for the different cases in (5.5). When we apply Lemma 2 to the bound (5.3) for $M \in [H, 5x^{1/4}]$, we find that

$$\begin{aligned}
S_2(H, 5x^{1/4}) &< h\left(\frac{1.2105\cdot 5^{7/3}x^{-1/12}}{1-1.045^{-7/3}} + \frac{0.0308x^{1/15}H^{-1/3}}{1-1.045^{-1/3}}\right) + \frac{1.7x^{1/20}}{1-1.045^{-1}} \\
&\quad + 0.5298x^{1/5}\left(\frac{\log\left(5x^{1/4}/H\right)}{\log(1.045)} + 1\right) \\
&< 0.1034h + 0.0001x^{1/5} + 0.5298x^{1/5}\left(\frac{\log x}{20\log(1.045)} - \frac{\log(12.1\log x)}{\log(1.045)} + 1\right) \\
&< 0.1034h + 0.6019x^{1/5}\log x - 89.7886x^{1/5} < 0.1582h.
\end{aligned}$$

Similarly, when we apply Lemma 2 to (5.3) for $M \in [5x^{1/4}, x^{2/5}]$, we get

$$\begin{aligned} S_2(5x^{1/4}, x^{2/5}) &< h\left(\frac{1.4182x^{-1/15}}{1-1.045^{-1/9}} + \frac{0.0308\cdot 5^{-1/3}x^{-1/60}}{1-1.045^{-1/3}}\right) + \frac{0.34x^{1/5}}{1-1.045^{-1}} \\ &\quad + 0.5298x^{1/5}\left(\frac{3\log x}{20\log(1.045)} - \frac{\log 5}{\log(1.045)} + 1\right) \\ &< 0.1148h + 1.8055x^{1/5}\log x - 10.9463x^{1/5} < 0.2790h. \end{aligned}$$

Hence,

$$S_2(H, x^{2/5}) < 0.1582h + 0.2790h = 0.4372h. \tag{6.1}$$

Next, we consider the case $H > 5x^{1/4}$ (which implies that $x \le e^{151}$). In this case we need only consider the latter case of Proposition 1 for M in the full range $(H, x^{2/5}]$. Applying Lemma 2 in this situation gives

$$\begin{aligned} S_2(H, x^{2/5}) &< h\left(\frac{1.4182x^{-1/15}}{1-1.045^{-1/9}} + \frac{0.0308x^{1/15}H^{-1/3}}{1-1.045^{-1/3}}\right) + \frac{0.34x^{1/5}}{1-1.045^{-1}} \\ &\quad + 0.5298x^{1/5}\left(\frac{\log x}{5\log(1.045)} - \frac{\log(60.5\log x)}{\log(1.045)} + 1\right) \\ &< 0.2378h + 2.4073x^{1/5}\log x - 98.1700x^{1/5} < 0.3976h, \end{aligned} \tag{6.2}$$

on noting that $98.170x^{1/5} > 0.059h$ when $x \le e^{151}$.

To complete the estimation of $S_2(H, \sqrt{2x})$, we apply Lemma 2 to the bound in Corollary 1 for $M \in [x^{2/5}, \sqrt{2x}]$. This yields

$$S_2(x^{2/5}, \sqrt{2x}) < \frac{0.0901x^{1/5}}{1-1.045^{-2}} + \frac{\log x}{10\log(1.045)} + \frac{0.5\log 2}{\log(1.045)} + 1 < 0.0009h. \tag{6.3}$$

Together, (6.1)–(6.3) establish (2.11) with

$$\sigma_3 = \begin{cases} 0.4381 & \text{if } H \le 5x^{1/4}, \\ 0.3985 & \text{if } H > 5x^{1/4}, \end{cases}$$

for all $x \ge e^{116}$. Taking $J = 120$ in (2.5), we have $\sigma_0(h, 120) \le -0.0595$ in the same range. Furthermore, for all $x \ge e^{116}$, we have $\sigma_2(h, 5.5) < 0.1797$, and for $x \ge e^{150}$, we have $\sigma_2(h, 5.5) < 0.1461$. Thus,

$$\sigma_0(h, 120) + \sigma_1 + \sigma_2(h, 5.5) + \sigma_3(h, 5.5) < \begin{cases} 0.9770 & \text{if } H \le 5x^{1/4}, \\ 0.9710 & \text{if } H > 5x^{1/4}, \end{cases}$$

which establishes (2.12), and therefore the theorem, for $x \ge e^{116}$.

6.2. **Intermediate x.** Suppose that $x \geq e^{41}$. We consider $h = 5x^{1/4}$ and choose $\lambda = 1.025$, $J = 19$ and $H = 1.75h$. With these choices, we apply Lemma 2 to the result of Corollary 3 to obtain

$$S_2(H, \sqrt{2x}) < \frac{1.4430 \cdot (0.025) x^{1/3} H^{-1/3}}{1 - 1.025^{-1/3}} + 2\left(\frac{\log\left(\sqrt{2x}\right) - \log H}{\log(1.025)} + 1\right)$$
$$< 0.4272h + \frac{\log x}{2\log(1.025)} + \frac{\log 2 - 2\log(8.75/1.025)}{\log(1.025)} < 0.4331h.$$

That is, (2.12) holds with $\sigma_3(h, 1.75) = 0.4331$. Moreover, when $h = 5x^{1/4}$ and $x \geq e^{41}$, we have

$$\sigma_0(h, 19) \leq -0.0543, \quad \sigma_2(h, 1.75) \leq 0.158.$$

Thus, when $h = 5x^{1/4}$ and $x \geq e^{41}$, we have

$$\sigma_0(h, 19) + \sigma_1 + \sigma_2(h, 1.75) + \sigma_3(h, 1.75) < 0.9891.$$

Together with the computations of [23], this proves the following result.

Proposition 3. *For any $x \geq 2$, the interval $(x, x + 5x^{1/4}]$ contains a squarefree integer.*

Moreover, an identical calculation for $x \geq e^{109}$ with $h = 3.8x^{1/4}$, $H = 4.5h$, $\lambda = 1.0001$, and $J = 100$ yields

$$\begin{aligned} &\sigma_0(h, 100) + \sigma_1 + \sigma_2(h, 4.5) + \sigma_3(h, 4.5) \\ &< -0.0594 + 0.4523 + 0.1571 + 0.4423 = 0.9924, \end{aligned}$$

which yields the following alternative.

Proposition 4. *For any $x \geq e^{109}$, the interval $(x, x + 3.8x^{1/4}]$ contains a squarefree integer.*

Since $5x^{1/4} \leq 11x^{1/5} \log x$ for $x \leq e^{109.7}$, Proposition 3 implies Theorem 1 for $x \leq e^{109}$. Finally, since $3.8x^{1/4} \leq 11x^{1/5} \log x$ for $x \leq e^{116.3}$, Proposition 4 establishes Theorem 1 when $e^{109} \leq x \leq e^{116}$. This completes the proof of the theorem.

7. Proof of Theorem 2

As stated in the introduction, the theorem can be checked by brute force for $x \leq e^{41}$. Thus, we focus on proving it for $x \geq e^{41}$.

7.1. **Large x.** Let $x \geq e^{200}$ and set $H = 11h$ in (2.2) and (2.11). We will use Proposition 2 and Lemma 2 to bound $S_3(H, x^{2/7})$.

Suppose first that $H \leq 18x^{1/6}$, (in this case we can assume $x \geq e^{284}$) we again split $S_3(H, x^{2/7})$ according to the two cases in (5.8). When we apply Lemma 2 to the bound (5.6) for $M \in [H, 18x^{1/6}]$, we find that

$$S_3(H, 18x^{1/6}) < h\left(\frac{0.9519 \cdot 18^{11/3} x^{-1/18}}{1 - 1.04^{-11/3}} + \frac{0.0374 x^{1/21} H^{-1/3}}{1 - 1.04^{-1/3}}\right) + \frac{3.8x^{1/42}}{1 - 1.04^{-1}}$$
$$+ 0.4212x^{1/7}\left(\frac{\log x}{42\log(1.04)} - \frac{\log(\frac{55}{18}\log x)}{\log(1.04)} + 1\right)$$
$$< 0.1552h + 0.2557x^{1/7}\log x - 72.2396x^{1/7} < 0.2064h.$$

Similarly, when we apply Lemma 2 to (5.6) for $M \in [18x^{1/6}, x^{2/7}]$, we get

$$S_3(18x^{1/6}, x^{2/7}) < h\left(\frac{5.1698x^{-1/21}}{1 - 1.04^{-1/15}} + \frac{0.0374 \cdot 18^{-1/3} x^{-1/126}}{1 - 1.04^{-1/3}}\right) + \frac{0.19x^{1/7}}{1 - 1.04^{-1}}$$
$$+ 0.4212x^{1/7}\left(\frac{5\log x}{42\log(1.04)} - \frac{\log 18}{\log(1.04)} + 1\right)$$
$$< 0.1180h + 1.2785x^{1/7}\log x - 25.6791x^{1/7} < 0.3737h.$$

Hence,

$$S_3(H, x^{2/7}) < 0.2064h + 0.3737h = 0.5801h. \tag{7.1}$$

Next, we consider the case $H > 18x^{1/6}$, noting that we must have $x \le e^{285}$. Since we need only use the latter case of Proposition 2 for M in the full range $(H, x^{2/7}]$ we find that

$$S_3(H, x^{2/7}) < h\left(\frac{5.1698x^{-1/21}}{1 - 1.04^{-1/15}} + \frac{0.0374x^{1/21}H^{-1/3}}{1 - 1.04^{-1/3}}\right) + \frac{0.19x^{1/7}}{1 - 1.04^{-1}}$$
$$+ 0.4212x^{1/7}\left(\frac{\log x}{7\log(1.04)} - \frac{\log(55\log x)}{\log(1.04)} + 1\right)$$
$$< 0.2742h + 1.5342x^{1/7}\log x - 94.5742x^{1/7} < 0.5148h, \tag{7.2}$$

on noting that $94.5742x^{1/7} > 0.0663h$ when $x \le e^{285}$.

To complete the estimation of $S_3(H, \sqrt[3]{2x})$, we apply Lemma 2 to the bound in Corollary 1 for $M \in [x^{2/7}, \sqrt[3]{2x}]$. This yields

$$|S_3(x^{2/7}, \sqrt[3]{2x})| \le \frac{0.1202x^{1/7}}{1 - 1.04^{-3}} + \frac{\log x}{21\log(1.04)} + \frac{\log 2}{3\log(1.04)} + 1 < 0.0011h. \tag{7.3}$$

Combining (7.1)–(7.3), we obtain (2.11) with

$$\sigma_3 = \begin{cases} 0.5812 & \text{if } H \le 18x^{1/6}, \\ 0.5159 & \text{if } H > 18x^{1/6}, \end{cases}$$

for all $x \ge e^{200}$. Taking $J = 100$ in (2.5), we have $\sigma_0(h, 100) \le -0.0066$ in the same range. Furthermore, we have

$$\sigma_2(h, 11) < \begin{cases} 0.2256 & \text{if } x \ge e^{284}, \\ 0.3020 & \text{if } x \ge e^{200}. \end{cases}$$

Thus,

$$\sigma_0(h,100)+\sigma_1+\sigma_2(h,11)+\sigma_3(h,11) < \begin{cases} 0.9750 & \text{if } H \le 18x^{1/6}, \\ 0.9861 & \text{if } H > 18x^{1/6}, \end{cases}$$

which establishes (2.12), and therefore the theorem, for $x \ge e^{200}$.

7.2. **Intermediate x.** Suppose that $x \ge e^{41}$. We take $h = 2x^{1/5}$, $H = 2.7h$, and $\lambda = 1.06$, and apply Lemma 2 to the result of Corollary 2. We obtain

$$\begin{aligned} S_3(H, \sqrt[3]{2x}) &\le \frac{1.2604(0.06)(xH^{-2})^{1/3}}{1-1.06^{-2/3}} + \frac{\log \sqrt[3]{2x} - \log H}{\log(1.06)} + 1 \\ &< 0.3225h + \frac{2\log x}{15\log(1.06)} - \frac{\log(5.4) - \frac{1}{3}\log 2}{\log(1.06)} + 1 < 0.3354h. \end{aligned}$$

That is, (2.11) holds with $\sigma_3(h,2.7) = 0.3354$. As $\sigma_2(h,2.7) \le 0.3146$, and $\sigma_0(h,6) < -0.0047$ we get

$$\sigma_0(h,6)+\sigma_1+\sigma_2(h,2.7)+\sigma_3(h,2.7) < 0.8201.$$

This establishes that the interval $(x, x+2x^{1/5}]$ contains a cubefree integer for all $x \ge e^{41}$. Moreover, recalling the results of [23], we obtain the following proposition.

Proposition 5. *For any $x \ge 2$, the interval $(x, x+2x^{1/5}]$ contains a cubefree integer.*

Since $2x^{1/5} \le 5x^{1/7}\log x$ for $x \le e^{95.8}$ the main result follows from Proposition 5 for $x \le e^{95}$. Next, we prove that $h = 10x^{1/6}$ is admissible when $x \ge e^{95}$. With this choice of h, we let $H = 4h$ and $\lambda = 1.03$. An application of Lemma 2 to the bound of Corollary 4 yields

$$S_3(H, \sqrt[3]{2x}) \le \frac{0.0726(xH^{-1})^{1/5}}{1-1.03^{-1/5}} + \frac{2\log x}{6\log(1.03)} < 0.5891h,$$

or $\sigma_3(h,4) = 0.5891$ in (2.11). Since $\sigma_0(h,20) < -0.0066$ and $\sigma_2(h,4) < 0.2207$ for $x \ge e^{95}$, we deduce that

$$\sigma_0(h,20)+\sigma_1+\sigma_2(h,4)+\sigma_3(h,4) < 0.9780.$$

Since $10x^{1/6} \le 5x^{1/7}\log x$ in the range $x \le e^{191.6}$, this establishes the theorem for $x \le e^{191}$.

By the same method we show that $h = 8.5x^{1/6}$ is admissible when $x \ge e^{191}$. With this choice of h, we let $H = 5h$ and $\lambda = 1.01$. In this case, applying Lemma 2 to the bound of Corollary 4 yields

$$S_3(H, \sqrt[3]{2x}) \le \frac{0.0242(xH^{-1})^{1/5}}{1-1.01^{-1/5}} + \frac{2\log x}{6\log(1.01)} < 0.6767h,$$

or $\sigma_3(h, 5) = 0.6767$ in (2.11). Since $\sigma_0(h, 20) < -0.0066$ and $\sigma_2(h, 5) < 0.1465$ for $x \geq e^{191}$, we deduce that

$$\sigma_0(h, 20) + \sigma_1 + \sigma_2(h, 5) + \sigma_3(h, 5) < 0.9914.$$

Since $8.5x^{1/6} \leq 5x^{1/7} \log x$ in the range $x \leq e^{200.3}$, this completes the proof of Theorem 2.

As we close this section, we take a moment to record the following result, which we just proved.

Proposition 6. *For any $x \geq e^{95}$, the interval $(x, x + 10x^{1/6}]$ contains a cubefree integer, and for any $x \geq e^{191}$, the interval $(x, x+8.5x^{1/6}]$ contains a cubefree integer.*

8. Asymptotic Results and Final Comments

We conclude by noting a few of the explicit bounds that can be obtained by these methods if one no longer requires the bounds to be admissible for all values of $x \geq 2$, allowing instead results valid for sufficiently large values of x.

Some of the possible results that can be obtained by tweaking the parameters used in the proof of Theorem 1 are given in the statement of Theorem 3. To prove any of those results, we reset the parameters m, J, λ, δ that appear in the proofs of Proposition 1 and Theorem 1 and then update the various constants. (When x is as large as in Theorem 3, the inequality $H \leq 5x^{1/4}$ always holds, so only the first case in the proof of Theorem 1 can occur.) To establish the claims of Theorem 3, we always select $J = 100$, $\lambda = 1.02$, and $m = \sqrt{\log x_0}$, where x_0 is the lower bound on x in each result; we only vary the choice of δ. For example, when $h = 5x^{1/5} \log x$, $x \geq e^{400}$ (hence, $m = 20$), and $\delta = 0.3$, we have

$$\sigma_0(h, 100) + \sigma_1 + \sigma_2(h, m) + \sigma_3(h, m) < 0.9811.$$

For $h = 2x^{1/5} \log x$ and $x \geq e^{1800}$, the choice $\delta = 0.6$ yields an upper bound of 0.9857; and for $h = x^{1/5} \log x$ and $x \geq e^{500\,000}$, $\delta = 0.87$ gives a bound of 0.9981.

We obtain Theorem 4 by making similar adjustments to the proofs of Proposition 2 and Theorem 2. In this case, we set $\lambda = 1.002$, $J = 20$, and $m = \sqrt{\log x_0}$, where x_0 is the lower bound on x, and again vary the choice of δ. For $h = 2x^{1/7} \log x$ and $x \geq e^{550}$, we set $\delta = 0.38$ to get a bound of 0.9914. For $h = x^{1/7} \log x$ and $x \geq e^{2300}$, we set $\delta = 0.66$, resulting in a bound of 0.9919. Finally, for $h = \frac{1}{2}x^{1/7} \log x$ and $x \geq e^{75\,000}$, we set $\delta = 0.90$ and get a bound of 0.9977.

Remark 2. Looking back at the proofs of our theorems, one can see that the value of h in our theorems is of the form $h(x) = b_k x^{1/(2k+1)} \log x$, with b_k an upper bound for a rather complicated bounded function $B_k(x; m, J, \lambda, \delta)$, which is decreasing in the variable x. Once x is sufficiently large, the decay in x appears to overwhelm the effect of the other parameters. On the other hand, to claim a specific value of b_k for all $x \geq x_0$, one generally needs to find acceptable choice of the other parameters to ensure that (2.12) holds. It seems that if one were to make the function $B_k(x; m, J, \lambda, \delta)$ fully explicit, one may even be able to identify a four-dimensional neighborhood of the chosen values of m, J, λ, δ such that all the choices of the parameters in that neighborhood are acceptable.

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References

[1] R. C. Baker and J. Pintz, *The distribution of squarefree numbers*, Acta Arith. **46** (1985), no. 1, 73–79. MR831264
[2] R. C. Baker and K. Powell, *The distribution of k-free numbers*, Acta Math. Hungar. **126** (2010), no. 1-2, 181–197. MR2593323
[3] P. Erdős, *Some problems and results in elementary number theory*, Publ. Math. Debrecen **2** (1951), 103–109. MR45759
[4] M. Filaseta, *An elementary approach to short interval results for k-free numbers*, J. Number Theory **30** (1988), no. 2, 208–225. MR961917
[5] ———, *Short interval results for squarefree numbers*, J. Number Theory **35** (1990), no. 2, 128–149. MR1057318
[6] M. Filaseta, S. W. Graham, and O. Trifonov, *Starting with gaps between k-free numbers*, Int. J. Number Theory **11** (2015), no. 5, 1411–1435. MR3376218
[7] M. Filaseta and O. Trifonov, *On gaps between squarefree numbers*, Analytic Number Theory (Allerton Park, IL, 1989), Progr. Math., vol. 85, Birkhäuser Boston, Boston, MA, 1990, pp. 235–253. MR1084183
[8] ———, *On gaps between squarefree numbers. II*, J. London Math. Soc. (2) **45** (1992), no. 2, 215–221. MR1171549
[9] ———, *The distribution of fractional parts with applications to gap results in number theory*, Proc. London Math. Soc. (3) **73** (1996), no. 2, 241–278. MR1397690
[10] E. Fogels, *On average values of arithmetic functions*, Proc. Cambridge Philos. Soc. **37** (1941), 358–372. MR4843
[11] S. W. Graham, *The distribution of squarefree numbers*, J. London Math. Soc. (2) **24** (1981), no. 1, 54–64. MR623670
[12] S. W. Graham and G. Kolesnik, *On the difference between consecutive squarefree integers*, Acta Arith. **49** (1988), no. 5, 435–447. MR967330
[13] A. Granville, *ABC allows us to count squarefrees*, Internat. Math. Res. Notices (1998), no. 19, 991–1009. MR1654759
[14] H. Halberstam and K. F. Roth, *On the gaps between consecutive k-free integers*, J. London Math. Soc. **26** (1951), 268–273. MR43120
[15] C. Hooley, *On the power free values of polynomials*, Mathematika **14** (1967), 21–26. MR214556
[16] M. N. Huxley and M. Nair, *Power free values of polynomials. III*, Proc. London Math. Soc. (3) **41** (1980), no. 1, 66–82. MR579717
[17] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, Dover Publications, Inc., New York, 1994, Corrected reprint of the 1966 original. MR1280462

[18] C.-H. Jia, *The distribution of square-free numbers*, Sci. China Ser. A **36** (1993), no. 2, 154–169. MR1223084

[19] H.-Q. Liu, *On the distribution of k-free integers*, Acta Math. Hungar. **144** (2014), no. 2, 269–284. MR3274401

[20] ———, *On the distribution of squarefree numbers*, J. Number Theory **159** (2016), 202–222. MR3412720

[21] L. Marmet, *First occurrences of square-free gaps and an algorithm for their computation*, preprint arXiv:1210.3829, 2012.

[22] H. L. Montgomery and R. C. Vaughan, *The distribution of squarefree numbers*, Recent Progress in Analytic Number Theory, Vol. 1 (Durham, 1979), Academic Press, London-New York, 1981, pp. 247–256. MR637350

[23] Michael J. Mossinghoff, Tomás Oliveira e Silva, and Timothy S. Trudgian, *The distribution of k-free numbers*, Math. Comp. **90** (2021), no. 328, 907–929. MR4194167

[24] M. Nair, *Power free values of polynomials. II*, Proc. London Math. Soc. (3) **38** (1979), no. 2, 353–368. MR531167

[25] R. A. Rankin, *Van der Corput's method and the theory of exponent pairs*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 147–153. MR72170

[26] H.-E. Richert, *On the difference between consecutive squarefree numbers*, J. London Math. Soc. **29** (1954), 16–20. MR57898

[27] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94. MR137689

[28] K. F. Roth, *On the gaps between squarefree numbers*, J. London Math. Soc. **26** (1951), 263–268. MR43119

[29] P. G. Schmidt, *Abschätzungen bei unsymmetrischen Gitterpunktproblemen*, Dissertation, Göttingen, 1964, Dissertation zur Erlangung des Doktorgrades der Mathematisch-Naturwissenschaftlichen Fakultät der Georg-August-Universität zu Göttingen. MR0181609

[30] O. Trifonov, *On the squarefree problem. II*, Math. Balkanica (N.S.) **3** (1989), no. 3-4, 284–295. MR1048051

[31] ———, *On gaps between k-free numbers*, J. Number Theory **55** (1995), no. 1, 46–59. MR1361558

[32] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. MR0220685

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