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# NOTE ON SETS WITHOUT GEOMETRIC PROGRESSIONS

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**Abstract**

For $k \geq 3$, we call a set $G \subseteq (0,1]$ of real numbers *k-good* if $G$ contains no geometric progression of length $k$ with integer ratio $r > 1$. A real number $x \in (0,1] \setminus G$ is called *k-bad with respect to G* if $G \bigcup \{x\}$ contains the $k$-term progression $(x, xr, xr^2, \cdots, xr^{k-1})$ for some integer $r > 1$. Define $Bad(G) = \{x \in (0,1] \setminus G :$ $x$ is $k$-bad with respect to $G\}$. In 2015, Nathanson and O'Bryant showed there exists a unique sequence of integers $\{1 = A_1^{(k)} < A_2^{(k)} < \cdots\}$ such that $G^{(k)} = \bigcup_{i=1}^{\infty}(1/A_{2i}^{(k)}, 1/A_{2i-1}^{(k)}]$ is a $k$-good set and $Bad(G^{(k)}) = \bigcup_{i=1}^{\infty}(1/A_{2i+1}^{(k)}, 1/A_{2i}^{(k)}]$. The values of $A_i^{(k)}$ for $2 \leq i \leq 5$ have previously been found by Nathanson, O'Bryant and the second author. In this note, we obtain the value of $A_6^{(k)}$ and pose a related problem.

## 1. Introduction

For an integer $k \geq 3$, we call a set $G \subseteq (0,1]$ of real numbers *k-good* if $G$ contains no geometric progression of length $k$ with integer ratio $r > 1$. A real number $x \in (0,1] \setminus G$ is called *k-bad with respect to G* if there exists an integer $r > 1$ such that $G \bigcup \{x\}$ contains the $k$-term geometric progression $(x, xr, xr^2, \cdots, xr^{k-1})$. Define

$$Bad(G) = \{x \in (0,1] \setminus G : x \text{ is } k\text{-bad with respect to } G\}.$$

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In 2015, Nathanson and O'Bryant [3] proved the following theorem.

**Theorem A** ([3]). *Fix $k \geq 3$. There exists a unique strictly increasing sequence of positive integers $\{A_1^{(k)} < A_2^{(k)} < \cdots\}$ with $A_1^{(k)} = 1$ such that*

$$G^{(k)} = \bigcup_{i=1}^{\infty} \Big( \frac{1}{A_{2i}^{(k)}}, \frac{1}{A_{2i-1}^{(k)}} \Big]$$

*is a k-good set and*

$$Bad(G^{(k)}) = \bigcup_{i=1}^{\infty} \Big( \frac{1}{A_{2i+1}^{(k)}}, \frac{1}{A_{2i}^{(k)}} \Big].$$

Nathanson and O'Bryant also proved in [3] that $A_2^{(k)} = 2^{k-1}$, $A_3^{(k)} = 2^k$ and

$$A_4^{(k)} = \begin{cases} 2^k 3^{k-l-1} & \text{if there is a positive integer } l \text{ such that } 2^{k-1} < 3^l < 2^k, \\ 3^{k-1} & \text{otherwise.} \end{cases}$$

Afterwards, the second author determined the value of $A_5^{(k)}$.

**Theorem B** ([1]). *Let $k \geq 3$ be an integer.*
*(i) If there is no integral power of 3 between $2^{k-1}$ and $2^k$, then*

$$A_5^{(k)} = 2^{k-1} 3^{i_0+1}, \text{ where } i_0 \text{ is the largest integer } i \text{ such that } 3^i < 3^{k-1}/2^k.$$

*(ii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$, then $k \geq 4$ and*

$$A_5^{(k)} = \begin{cases} 200 & \text{if } k = 4, \\ 2^{k-1} 3^{k-l} & \text{if } k \geq 5. \end{cases}$$

For the remainder of the paper we use the variable $l$ for the least integer such that $3^l > 2^{k-1}$. This convention allows for a simpler statement of the above theorem. The reader can check that case (i) above simplifies to the second case, and we get the following, valid for all $k \geq 3$:

$$A_5^{(k)} = \begin{cases} 200 & \text{if } k = 4, \\ 2^{k-1} 3^{k-l} = 3^{k-1} \left( \frac{2^{k-1}}{3^{l-1}} \right) & \text{otherwise.} \end{cases}$$

Note that the value $\frac{1}{A_4^{(k)}}$ is the upper limit of an interval of values that are bad with respect to the set $\left(1/A_4^{(k)}, 1/A_3^{(k)}\right] \bigcup \left(1/A_2^{(k)}, 1\right]$ (write this set as $G_2^{(k)}$) because of progressions with ratio $r = 3$. The ratio $\left(\frac{2^{k-1}}{3^{l-1}}\right)$ is the ratio between $\frac{1}{3^{l-1}}$ and $\frac{1}{2^{k-1}}$, the largest value that is excluded from $G_2^{(k)}$. When $x = \frac{1}{2^{k-1} 3^{k-l}} = 1/A_5^{(k)}$, the term $3^{k-l} x = \frac{1}{2^{k-1}}$, and so the entire progression with ratio 3 starting at $x$ is no longer contained in $G_2^k$.

In this note, we obtain the value of $A_6^{(k)}$.

**Theorem 1.** *Let $k \geq 3$ be an integer.*
*(i) If there is no integral power of* 3 *between $2^{k-1}$ and $2^k$, then*

$$A_6^{(k)} = 2^k 3^{k-l}, \text{ where } l \text{ is the smallest integer such that } 3^l > 2^k.$$

*(ii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is no integral power of* 4 *between $4 \cdot 3^{k-l-1}$ and $2 \cdot 3^{k-l}$, then*

$$A_6^{(k)} = \begin{cases} 4^{k-1} & \text{for even } k, \\ 2^{2k-1} & \text{for odd } k. \end{cases}$$

*(iii) If there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is a positive integer $m$ such that $4 \cdot 3^{k-l-1} < 4^m < 2 \cdot 3^{k-l}$, then $A_6^{(4)} = 216$ and*

$$A_6^{(k)} = \begin{cases} \frac{1}{2} 4^{k-m} 3^{k-l} & \text{for even } k > 4, \\ 4^{k-m} 3^{k-l} & \text{for odd } k. \end{cases}$$

Based on the above results, we pose the following problem.

**Problem.** For each positive integer $i$, do there exist infinitely many positive integers $k$ such that $A_{2i}^{(k)} = (i+1)^{k-1}$?

One may refer to [2], [4] and [5] for related results.

## 2. Proof of Theorem 1

We first prove[2] that $A_6^{(3)} = 24$ and $A_6^{(4)} = 216$. For $k = 3$, we know that $A_2^{(k)} = 4$, $A_3^{(k)} = 8$, $A_4^{(k)} = 9$ and $A_5^{(k)} = 12$. For $x \in (\frac{1}{16}, \frac{1}{12}]$, we have $\frac{1}{8} < 2x < \frac{1}{4}$, $\frac{1}{8} < 3x \leq \frac{1}{4}$ and $r^2 x > 1$ for $r \geq 4$. So the set $\left(\frac{1}{16}, \frac{1}{12}\right] \bigcup \left(\frac{1}{9}, \frac{1}{8}\right] \bigcup \left(\frac{1}{4}, 1\right]$ is 3-good. Furthermore, for $x \in (\frac{1}{24}, \frac{1}{16}]$, we have $\frac{1}{8} < 2^2 x = 4x \leq \frac{1}{4}$, $\frac{1}{8} < 3x < \frac{1}{4}$ and $r^2 x > 1$ for $r \geq 5$, and so the set

$$\left(\frac{1}{24}, \frac{1}{12}\right] \bigcup \left(\frac{1}{9}, \frac{1}{8}\right] \bigcup \left(\frac{1}{4}, 1\right] \tag{1}$$

is also 3-good. However, for $x_0 = \frac{1}{24}$, we have $3x_0 = \frac{1}{8}$ and $\frac{1}{4} < 3^2 x_0 < 1$. Thus $x_0$ is 3-bad with respect to the set (1), and so $A_6^{(3)} = 24$.

For $k = 4$, we know $A_2^{(k)} = 8$, $A_3^{(k)} = 16$, $A_4^{(k)} = 48$ and $A_5^{(k)} = 200$. For $x \in (\frac{1}{216}, \frac{1}{200}]$, we have that $\frac{1}{200} < 3x < 2^2 x = 4x \leq \frac{1}{48}$, while $\frac{1}{16} < 5^2 x \leq \frac{1}{8}$ and $r^3 x > 1$ for $r \geq 6$. When $x_0 = \frac{1}{216}$, we have $\frac{1}{48} < 6x_0 < \frac{1}{16}$ and $\frac{1}{8} < 6^2 x_0 < 6^3 x_0 = 1$. Thus,

$$\left(\frac{1}{216}, \frac{1}{200}\right] \bigcup \left(\frac{1}{48}, \frac{1}{16}\right] \bigcup \left(\frac{1}{8}, 1\right]$$

[2]These values were included in [3], but we include a proof for completeness.

is 4-good and $A_6^{(4)} = 216$. In the following we can assume $k \geq 5$. We divide the proof into three cases.

`Case 1.` Assume there is no integral power of 3 between $2^{k-1}$ and $2^k$. In this case, $A_4^{(k)} = 3^{k-1}$, $A_5^{(k)} = 2^{k-1}3^{k-l}$ and $3^{l-1} < 2^{k-1} < 2^k < 3^l$. Let

$$G_2^{(k)} = (\frac{1}{3^{k-1}}, \frac{1}{2^k}] \bigcup (\frac{1}{2^{k-1}}, 1].$$

For $\frac{1}{2^k 3^{k-l}} < x \leq \frac{1}{2^{k-1}3^{k-l}}$, and any $r \geq 4$, we have

$$r^{k-1}x \geq 4^{k-1}x > \frac{4^{k-1}}{2^k 3^{k-l}} > \frac{4^{k-1}}{3^k} > 1. \tag{2}$$

Note the last inequality follows since $k \geq 5$. When $r = 3$, we have $\frac{1}{2^k} < 3^{k-l}x \leq \frac{1}{2^{k-1}}$, so $3^{k-l}x \notin G_2^{(k)}$. Finally, for $r = 2$, note that after dividing through (2) by $2^{k-1}$ we find that $2^{k-1}x > \frac{1}{2^{k-1}}$. Thus, there must exist an integer $m \leq k-1$ such that $\frac{1}{2^k} < 2^m x \leq \frac{1}{2^{k-1}}$ and so $2^m x \notin G_2^{(k)}$.

Thus $(\frac{1}{2^k 3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}] \bigcup G_2^{(k)}$ is $k$-good. Now we prove that $x_0 = \frac{1}{2^k 3^{k-l}}$ is $k$-bad with respect to $G_2^{(k)}$. Using again that $3^{l-1} < 2^k < 3^l$,

$$\frac{1}{3^{k-1}} = \frac{2^k}{2^k 3^{k-1}} < \frac{1}{2^k 3^{k-l-1}} = 3x_0 < 3^2 x_0 < \cdots < 3^{k-l}x_0 = \frac{1}{2^k}$$

and then

$$\frac{1}{2^{k-1}} < 3^{k-l+1}x_0 < \cdots < 3^{k-1}x_0 = \frac{3^{k-1}}{2^k 3^{k-l}} = \frac{3^{l-1}}{2^k} < 1.$$

Thus,

$$\{3^i x_0 : i = 1, 2, \cdots, k-1\} \subseteq G_2^{(k)}.$$

That is, $x_0$ is $k$-bad with respect to $G_2^{(k)}$. To sum up, in Case 1, we have shown that $A_6^{(k)} = 2^k 3^{k-l}$.

For the remainder of the proof we can assume there is an integer $l$ with $2^{k-1} < 3^l < 2^k$. Thus, $A_4^{(k)} = 2^k 3^{k-l-1}$ and $A_5^{(k)} = 2^{k-1}3^{k-l}$. For the remainder we set

$$G_2^{(k)} = (\frac{1}{2^k 3^{k-l-1}}, \frac{1}{2^k}] \bigcup (\frac{1}{2^{k-1}}, 1].$$

We first prove the following lemma which will be used throughout to handle the ratio $r = 3$.

**Lemma 1.** *Suppose that $k \geq 3$ and there exists an integer $l$ with $2^{k-1} < 3^l < 2^k$. Then for every $x$ in the interval $\frac{1}{3^{k-1}2^k} < x \leq \frac{1}{2^{k-1}3^{k-l}} = \frac{1}{A_5^{(k)}}$, there exists an integer $1 \leq n \leq k-1$ such that either $\frac{1}{A_5^{(k)}} < 3^n x \leq \frac{1}{A_4^{(k)}}$ or $\frac{1}{A_3^{(k)}} < 3^n x \leq \frac{1}{A_2^{(k)}}$.*

*Proof.* For a fixed value of $x$, let $n_1$ be the smallest integer such that $\frac{1}{A_5^{(k)}} = \frac{1}{2^{k-1}3^{k-l}} < 3^{n_1}x$, and $n_2$ the smallest integer such that $\frac{1}{A_3^{(k)}} = \frac{1}{2^k} < 3^{n_2}x$. By $3^{k-1}x > \frac{1}{2^k}$, we have $0 < n_1 \le n_2 \le k-1$. Suppose that $3^{n_2}x \notin \left(\frac{1}{A_3^{(k)}}, \frac{1}{A_2^{(k)}}\right]$. Then

$$3^{n_2}x > \frac{1}{2^{k-1}} > \frac{1}{2^k} \ge 3^{n_2-1}x.$$

Multiplying through by $\frac{1}{3^{k-l}}$ gives

$$3^{n_2-k+l}x > \frac{1}{2^{k-1}3^{k-l}} \ge 3^{n_2-k+l-1}x.$$

So $n_1 = n_2 - k + l$. On the other hand, multiplying through the same inequality by $\frac{1}{3^{k-l-1}}$ gives

$$\frac{1}{A_4^{(k)}} = \frac{1}{2^k 3^{k-l-1}} \ge 3^{n_2-k+l}x = 3^{n_1}x.$$

Thus, $3^{n_1}x \in \left(\frac{1}{A_5^{(k)}}, \frac{1}{A_4^{(k)}}\right]$. □

We now return to the proof of Theorem 1.1.

`Case 2.` Assume there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and there is no integral power of 4 between $4 \cdot 3^{k-l-1}$ and $2 \cdot 3^{k-l}$.

We consider the interval $\frac{1}{4^{k-1}} < x \le \frac{1}{2^{k-1}3^{k-l}}$. If $r \ge 4$, we have $r^{k-1}x \ge 4^{k-1}x > 1$. If $r = 3$ we apply Lemma 1. Since $\frac{1}{4^{k-1}} > \frac{1}{3^{k-1}2^k}$ it implies that

$$3^n x \in \left(\frac{1}{A_5^{(k)}}, \frac{1}{A_4^{(k)}}\right] \bigcup \left(\frac{1}{A_3^{(k)}}, \frac{1}{A_2^{(k)}}\right] \quad \text{for some } \ 1 \le n \le k-1.$$

For the remaining ratio $r = 2$, note that $2^{k-1}x > \frac{1}{2^{k-1}}$, so there exists some integer $0 < h < k-1$ such that $\frac{1}{2^k} < 2^h x \le \frac{1}{2^{k-1}}$. Thus the set

$$G_3^{(k)} := \left(\frac{1}{4^{k-1}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$$

is $k$-good.

We now suppose further that $k$ is even and we will prove that $x_0 = \frac{1}{4^{k-1}}$ is $k$-bad with respect to $G_3^{(k)}$ by showing that for each $1 \le i \le k-1$, the term $4^i x_0 = \frac{1}{4^{k-i-1}} \in G_3^{(k)}$. When $i = k-1$ we have $4^{k-1}x_0 = 1$. Since $k$ is even, $k = 2j$ for some integer $j$ and $4^{k-j-1}x_0 = \frac{1}{4^j} = \frac{1}{2^k} = \frac{1}{A_3^{(k)}} \in G_3^{(k)}$. Clearly there is no power of 4 strictly between $2^k$ and $2^{k-1}$, so it suffices to show that none of the terms $4^i x_0$ fall in the gap between $\frac{1}{A_5^{(k)}}$ and $\frac{1}{A_4^{(k)}}$, i.e., there is no $i$ with

$$\frac{1}{2^{k-1}3^{k-l}} < \frac{1}{4^{k-i-1}} \le \frac{1}{2^k 3^{k-l-1}}. \tag{3}$$

If there were such an $i$, multiplying through (3) by $2^{k-2} = 4^{j-1}$ we get

$$\frac{1}{2\cdot 3^{k-l}} < \frac{1}{4^{k-i-j}} \leq \frac{1}{2^2 3^{k-l-1}},$$

which contradicts the assumption that there is no integral power of 4 between $4\cdot 3^{k-l-1}$ and $2\cdot 3^{k-l}$. Thus, when $k$ is even we have $A_6^{(k)} = \frac{1}{4^{k-1}}$.

We now consider odd $k$. The argument above for even $k$ implies that $G_3^{(k)}$ is still $k$-good and we will show, in this case, that the larger set $(\frac{1}{2\cdot 4^{k-1}}, \frac{1}{4^{k-1}}] \bigcup G_3^{(k)}$ is also $k$-good. For $\frac{1}{2\cdot 4^{k-1}} < x \leq \frac{1}{4^{k-1}}$, if $r \geq 5$, then

$$r^{k-1}x \geq 5^{k-1}x > \frac{1}{2}\cdot(\frac{5}{4})^{k-1} > 1.$$

If $r = 4$, using that $k$ is odd and noting that $\frac{1}{2^k} < 4^{\frac{k-1}{2}}x \leq \frac{1}{2^{k-1}}$, we know that $4^{\frac{k-1}{2}}x \notin G_2^{(k)}$. When $r = 3$, Lemma 1 again shows there exists $1 \leq n \leq k-1$ with $3^n x \notin (\frac{1}{2\cdot 4^{k-1}}, \frac{1}{4^{k-1}}] \bigcup G_2^{(k)}$. Finally, for $r = 2$, we have $\frac{1}{2^k} < 2^{k-1}x \leq \frac{1}{2^{k-1}}$, and so we know that $2^{k-1}x \notin G_2^{(k)}$. Thus $\left(\frac{1}{2\cdot 4^{k-1}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ is $k$-good.

To finish this case we show that $x_0 = \frac{1}{2\cdot 4^{k-1}}$ is $k$-bad with respect to this set by showing that $4^i x_0$ is contained in it for each $1 \leq i \leq k-1$. When $i = k-1$ we have $4^{k-1}x_0 = \frac{1}{2} > \frac{1}{A_2^{(k)}}$. Since $k$ is odd, $k = 2j+1$ for some integer $j$. Now $4^{k-j-1}x_0 = \frac{1}{2\cdot 4^j} = \frac{1}{2^k} = \frac{1}{A_3^{(k)}}$ and there is no power of 4 strictly between $2^k$ and $2^{k-1}$, so it remains to show that there is no $i$ with

$$\frac{1}{2^{k-1}3^{k-l}} < \frac{1}{2\cdot 4^{k-i-1}} \leq \frac{1}{2^k 3^{k-l-1}}.$$

If there were, multiplying through this time by $2^{k-2} = 2\cdot 4^{j-1}$ gives

$$\frac{1}{2\cdot 3^{k-l}} < \frac{1}{4^{k-i-j}} \leq \frac{1}{2^2 3^{k-l-1}},$$

contradicting the assumption there is no integral power of 4 between $4\cdot 3^{k-l-1}$ and $2\cdot 3^{k-l}$. To sum up, in Case 2, we have shown that

$$A_6^{(k)} = \begin{cases} 4^{k-1} & \text{for even } k, \\ 2^{2k-1} & \text{for odd } k. \end{cases}$$

`Case 3.` Assume that there is a positive integer $l$ such that $2^{k-1} < 3^l < 2^k$ and a positive integer $m$ such that

$$4\cdot 3^{k-l-1} < 4^m < 2\cdot 3^{k-l}. \tag{4}$$

The same argument used in Case 2 already shows that $\left(\frac{1}{4^{k-1}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ is $k$-good for even $k$ and that $\left(\frac{1}{2\cdot 4^{k-1}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ is $k$-good when $k$ is odd.

First, we consider even $k$ and the interval $\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}} < x \leq \frac{1}{4^{k-1}}$. Note, since $4^m > 4 \cdot 3^{k-l-1}$, that the lower bound $\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}} > \frac{2}{3 \cdot 4^{k-1}}$. If $r \geq 5$ then, since $k \geq 3$,

$$r^{k-1}x \geq 5^{k-1}x > \frac{5^{k-1}}{\frac{1}{2}4^{k-m}3^{k-l}} > \frac{8 \cdot 5^{k-1}}{3 \cdot 4^k} > 1.$$

For the ratio $r = 4$ we consider $4^{\frac{k}{2}-m}x$. Using the inequalities in (4) we deduce that

$$\frac{1}{2^{k-1}3^{k-l}} < 4^{\frac{k}{2}-m}x \leq \frac{1}{2^k 4^{m-1}} < \frac{1}{2^k 3^{k-l-1}} \tag{5}$$

and so $\frac{1}{A_5^{(k)}} < 4^{\frac{k}{2}-m}x < \frac{1}{A_4^{(k)}}$. Also, note that $1 \leq \frac{k}{2} - m < \frac{k}{2} - 1$, so $4^{\frac{k}{2}-m} = 2^{k-2m} < 2^{k-1}$, thus this observation handles the case of $r = 2$ as well. Finally, the ratio $r = 3$ can again be handled using Lemma 1, noting that

$$\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}} > \frac{2}{3 \cdot 4^{k-1}} > \frac{1}{3^{k-1}2^k}.$$

Thus, $\left(\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ is $k$-good.

We now show $x_0 = \frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}$ is $k$-bad with respect to $\left(\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$. We have

$$\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}4^{\frac{k}{2}-m} = \frac{1}{2^{k-1}3^{k-l}} \quad \text{and} \quad \frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}4^{\frac{k}{2}-m+1} = \frac{4}{2^{k-1}3^{k-l}} > \frac{1}{2^k 3^{k-l-1}}.$$

Furthermore, using (4) we find that

$$\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}4^{\frac{k}{2}-1} = \frac{4^m}{2^{k+1}3^{k-l}} < \frac{1}{2^k}, \quad \frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}4^{\frac{k}{2}} = \frac{4^m}{2^{k-1}3^{k-l}} > \frac{1}{2^{k-1}},$$

and $\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}4^{k-1} = \frac{4^m}{2 \cdot 3^{k-l}} < 1$. That is, $4^i x_0 \in \left(\frac{1}{\frac{1}{2}4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ for $1 \leq i \leq k-1$, so $x_0$ is $k$-bad with respect to that set. So $A_6^{(k)} = \frac{1}{2}4^{k-m}3^{k-l}$.

Now consider odd $k$. For $\frac{1}{2 \cdot 4^{k-1}} < x \leq \frac{1}{2^{k-1}3^{k-l}}$, by the same discussion as in the odd-$k$ part of Case 2, we know that $x$ is $k$-good with respect to $G_2^{(k)}$, so we take $\frac{1}{4^{k-m}3^{k-l}} < x \leq \frac{1}{2 \cdot 4^{k-1}}$. If $r \geq 5$, then

$$r^{k-1}x \geq 5^{k-1}x > 5^{k-1}\frac{1}{4^{k-m}3^{k-l}} > 1.$$

For $r = 2, 4$, it follows from (4) that

$$\begin{aligned} 4^{\frac{k+1}{2}-m}x = 2^{k-2m+1}x &> \frac{1}{4^{k-m}3^{k-l}}2^{k-2m+1} = \frac{1}{2^{k-1}3^{k-l}} \\ 4^{\frac{k+1}{2}-m}x = 2^{k-2m+1}x &\leq \frac{1}{2 \cdot 4^{k-1}}2^{k-2m+1} < \frac{1}{2^k 3^{k-l-1}}. \end{aligned}$$

Thus,

$$2^{k-2m+1}x = 4^{\frac{k+1}{2}-m}x \in \left(\frac{1}{A_5^{(k)}}, \frac{1}{A_4^{(k)}}\right].$$

For the remaining ratio, $r = 3$, we appeal one last time to Lemma 1. Since $\frac{1}{4^{k-m}3^{k-l}} > \frac{1}{3\cdot 4^{k-1}} > \frac{1}{3^{k-1}2^k}$ we are guaranteed the existence of an $n$, $1 \le n \le k-1$ with $3^n x \notin \left(\frac{1}{4^{k-m}3^{k-l}}, \frac{1}{4^{k-1}}\right] \bigcup G_2^{(k)}$.

Thus, $\left(\frac{1}{4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ is $k$-good. We conclude by proving that $x_0 = \frac{1}{4^{k-m}3^{k-l}}$ is $k$-bad with respect to $\left(\frac{1}{4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ for odd $k$. It follows from (4) that

$$\frac{1}{4^{k-m}3^{k-l}}2^{k-2m+1} = \frac{1}{2^{k-1}3^{k-l}}, \quad \frac{1}{4^{k-m}3^{k-l}}2^{k-2m+2} = \frac{2}{2^{k-1}3^{k-l}} > \frac{1}{2^k 3^{k-l-1}}$$

and

$$\frac{1}{4^{k-m}3^{k-l}}2^{k-1} = \frac{4^m}{2^{k+1}3^{k-l}} < \frac{1}{2^k}.$$

This means that $2^i x_0 \in \left(\frac{1}{4^{k-m}3^{k-l}}, \frac{1}{2^{k-1}3^{k-l}}\right] \bigcup G_2^{(k)}$ for each $1 \le i \le k-1$, so $x_0$ is $k$-bad with respect to that set. To sum up, in Case 3, we have shown that

$$A_6^{(k)} = \begin{cases} \frac{1}{2}4^{k-m}3^{k-l} & \text{for even } \ k > 4, \\ 4^{k-m}3^{k-l} & \text{for odd } \ k. \end{cases}$$

This completes the proof of Theorem 1. □

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