# Counting permutations and primitive sets
## using the divisor graph of $\{1, 2, \ldots n\}$

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# Coprime Mappings

Around 1960, D. J. Newman conjectured that for any interval, $I$ of $n$ consecutive integers, there is always a bijection $\psi : \{1, 2, \ldots, n\} \to I$ where $i$ and $\psi(i)$ are coprime for every integer $i$.

**Example:** $n = 6$, $I = \{25, 26, 27, 28, 29, 30\}$.

$$1 \to 30,\ 2 \to 25,\ 3 \to 26,\ 4 \to 27,\ 5 \to 28,\ 6 \to 29.$$

In 1962 Paul Erdős offered £5 for a proof of the weaker conjecture where $I = \{n + 1, ..., 2n\}$. A year later, D. E. Daykin and M. J. Baines proved the weaker conjecture.

In 1979 Pomerance and Selfridge proved the full Newman conjecture.

# Coprime permutations

If $I = \{1, 2, \ldots, n\}$, bijective functions $\psi$ are permutations of $n$.

Recently, Pomerance revisited this problem, asking how many permutations of $n$ have this coprime property. Call this count $C(n)$.

**Example:** $n = 4$: 2341 2143 4123 4321 C(4)=4.

He proved that $\frac{n!}{3.73^n} \leq C(n) \leq \frac{n!}{2.5^n}$ (for sufficiently large $n$).

I conjectured $C(n) = \frac{n!}{(c+o(1))^n}$, $c = \prod_p \frac{p(p-2)^{1-2/p}}{(p-1)^{2-2/p}} = 2.6504\ldots$

...two days later this was proved by Sah and Sawhney.

# Heuristic: Why $c = \prod_p \frac{p(p-2)^{1-2/p}}{(p-1)^{2-2/p}} = 2.6504...?$

Fix a prime $p$. Count permutations $\sigma$ of $n$ where $p \nmid \gcd(j, \sigma(j))$ for all $j$. (For simplicity, suppose $p|n$.)

First assign numbers coprime to $p$ to each value of $j$ divisible by $p$. There are $\binom{n-n/p}{n/p} \cdot \left(\frac{n}{p}\right)! = \left(\frac{p-1}{p}n\right)!/\left(\frac{p-2}{p}n\right)!$ ways to do that.

Then distribute the remaining $\frac{p-1}{p}n$ integers in any order. $(\frac{p-1}{p}n)!$ Multiplying, this gives

$$\frac{\left(\frac{p-1}{p}n\right)!^2}{\left(\frac{p-2}{p}n\right)!} \approx \frac{n!}{p^n}\frac{(p-1)^{\frac{2p-2}{p}n}}{(p-2)^{\frac{p-2}{p}n}} = n!\left(\frac{p-1}{p}\left(\frac{p-1}{p-2}\right)^{\frac{p-2}{p}}\right)^n$$

neglecting some constant terms.

# Heuristic: Why $c = \prod_p \frac{p(p-2)^{1-2/p}}{(p-1)^{2-2/p}} = 2.6504...$?

$$\frac{\left(\frac{p-1}{p}n\right)!^2}{\left(\frac{p-2}{p}n\right)!} \approx \frac{n!}{p^n}\frac{(p-1)^{\frac{2p-2}{p}n}}{(p-2)^{\frac{p-2}{p}n}} = n!\left(\frac{p-1}{p}\left(\frac{p-1}{p-2}\right)^{\frac{p-2}{p}}\right)^n$$

So, a random permutation has this property with probability

$$\left(\frac{p-1}{p}\left(\frac{p-1}{p-2}\right)^{\frac{p-2}{p}}\right)^n.$$

Finally, assume independence for different primes, and multiply all of these probabilities to get the probability that a permutation has this property for every prime $p$.

# More permutations

In the same paper, Pomerance raises the question of counting permutations $\sigma$ where for every index either $\sigma(i)|i$ or $i|\sigma(i)$.

**Example:** $n = 4$, 3412 1234 1432 2431 2134 3214 4231 4132

8 total.

Denote the count of these permutations by $D(n)$.

In a subsequent paper this year he proves that

$$1.93^n \leq D(n) \leq 13.6^n$$

for sufficiently large $n$.

# Main Theorem

**Theorem**

*As $n \to \infty$ we have*

$$D(n) = (c_d + o(1))^n$$

*where* $2.069 < c_d < 2.694$.

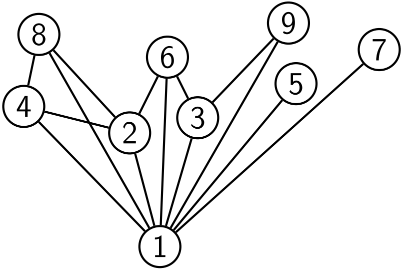
In fact we get a significantly better error term. For any $\epsilon > 0$ we have

$$D(n) = c_d^{n\left(1+O\left(\exp((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)}.$$

# The divisor graph

The **divisor graph** of $\{1, 2, \ldots, n\}$, denoted $\mathcal{D}_{[1,n]}$, is the graph on vertices $v_1, v_2, \ldots v_n$ and an edge between $v_i$ and $v_j$ if $i \mid j$.

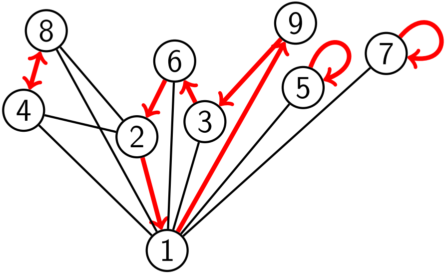
**Example:** $n = 9$



# The connection

$$\left\{\begin{array}{l}\text{Permutations } \sigma \text{ of } n \text{ where} \\ \forall\, i \text{ either } \sigma(i) \mid i \text{ or } i \mid \sigma(i)\end{array}\right\} \iff \left\{\begin{array}{l}\text{Directed, vertex-disjoint cycle} \\ \text{covers of } \mathcal{D}_{[1,n]} \text{ with self-loops}\end{array}\right\}$$

**Example:** $\sigma = 916852743 = (1\,9\,3\,6\,2)(4\,8)(5)(7)$



# Strategy

**General strategy for computing statistics on divisor graphs**

Write your statistic as a telescoping sum (or product) working "backward" from $n$.

Define

$$d(i, n) = \frac{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[i,n]}}{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[i+1,n]}}$$

so that $D(n) = \prod_{i=1}^{n} d(i, n)$.

Then group together like terms in this product.

# Another problem: counting primitive Sets

A set of integers is **primitive** if no integer in the set divides another.

**Examples:**

- Primes
- Integers with exactly $k$ prime factors counted with multiplicity.
- $\left\{\left\lfloor\frac{n}{2}\right\rfloor+1,\left\lfloor\frac{n}{2}\right\rfloor+2,\ldots,n\right\}$
- **Independent** subsets of the vertices of $\mathcal{D}_{[1,n]}$ are primitive sets

An **independent** set of vertices in a graph has no adjacent vertices.

# Finite primitive sets

Note: $\{\}$ and $\{1\}$ are both primitive sets.
What do other primitive subsets of the first few integers look like?

| $n$ | primitive subsets of $\{1, 2 \ldots n\}$ | count |
|---|---|---|
| 1 | $\{\}$, $\{1\}$ | 2 |
| 2 | $\{\}$, $\{1\}$, $\{2\}$ | 3 |
| 3 | $\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$ | 5 |
| 4 | $\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$ | 7 |
| 5 | $\{\}$, $\{1\}$,$\{2\}$,$\{3\}$,$\{2, 3\}$, $\{4\}$,$\{3, 4\}$,$\{5\}$,$\{2, 5\}$,$\{3, 5\}$, $\{2, 3, 5\}$, $\{4, 5\}$,$\{3, 4, 5\}$ | 13 |

How many primitive subsets of $\{1, 2 \ldots n\}$ are there?

# Counting primitive sets

Let $Q(n)$ count the primitive sets with largest element at most $n$.

A051026 Number of primitive subsequences of {1, 2, ..., n}. 5

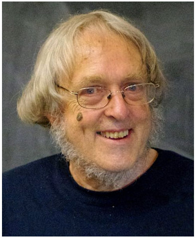
1, 2, 3, 5, 7, 13, 17, 33, 45, 73, 103, 205, 253, 505, 733, 1133, 1529, 3057, 3897, 7793, 10241, 16513, 24593, 49185, 59265, 109297, 163369, 262489, 355729, 711457, 879937, 1759873, 2360641, 3908545, 5858113, 10534337, 12701537, 25403073, 38090337, 63299265, 81044097, 162088193, 205482593, 410965185, 570487233, 855676353 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS a(n) counts all subsequences of {1, ..., n} in which no term divides any other. If n is a prime a(n) = 2*a(n-1)-1 because for each subsequence s counted by a(n-1) two different subsequences are counted by a(n): s and s,n. There is only one exception: 1,n is not a primitive subsequence because 1 divides n. For all n>1: a(n) < 2*a(n-1). - Alois P. Heinz, Mar 07 2011

# Bounds

Let $Q(n)$ count the primitive sets with largest element at most $n$. Cameron and Erdős considered $Q(n)$ in 1990.



(Almost) trivially:

$$(\sqrt{2})^n < Q(n) < 2^n$$

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive. There are $2^{\lceil \frac{n}{2} \rceil} > \sqrt{2}^n$ such subsets.

# Bounds

Cameron and Erdős (1990) improve these bounds to:

$$1.5596^n < Q(n) < 1.60^n$$

**Conjecture:** $\lim_{n\to\infty} Q(n)^{1/n}$ exists.

**Theorem (Angelo, 2017)**

*The limit* $\lim_{n\to\infty} Q(n)^{1/n}$ *exists. Equivalently,* $Q(n) = c^{n+o(1)}$*.*

Proof is not effective: Gives no insight on the value of this constant.

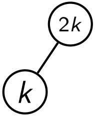
# Constructing primitive sets using the divisor graph

Every subset of $\left(\frac{n}{2}, n\right]$ is primitive, so $Q(n) \geq 2^{n/2} \approx 1.4142^n$.

Note that $\mathcal{D}_{\left(\frac{n}{2}, n\right]}$ consists only of isolated vertices.

Now take $k \in \left(\frac{n}{3}, \frac{n}{2}\right] \subset \mathcal{D}_{\left(\frac{n}{3}, n\right]}$. Include $k, 2k$ or neither. (3 possibilities)

Equivalent to counting independent subsets of the graph:




Use this to improve the lower bound for $Q(n)$.

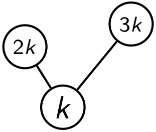
$Q(n) \geq 2^{n/2}\left(\frac{3}{2}\right)^{n/6} = 2^{n/3}3^{n/6} \approx 1.5131^n$.

# Constructing primitive sets: pressing on!

Work backward, considering progressively smaller $k$. Take $k \in \left(\frac{n}{4}, \frac{n}{3}\right]$.

Now we need to worry about $k, 2k, 3k$. Actually two scenarios:

**Odd** $k$:

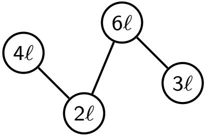



5 primitive subsets of $\{k, 2k, 3k\}$ replace 4 of just $\{2k, 3k\}$.

**Even** $k = 2\ell$: Now $\frac{3k}{2} = 3\ell$ is also an integer.

Consider subsets of $k{=}2\ell$, $3\ell$, $2k{=}4\ell$, $3k{=}6\ell$.

Find 8 primitive subsets:
$\{\}$, $\{2\ell\}$, $\{3\ell\}$, $\{4\ell\}$, $\{6\ell\}$,
$\{2\ell, 3\ell\}$, $\{3\ell, 4\ell\}$, $\{4\ell, 6\ell\}$



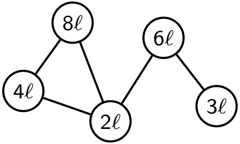

Replacing the 6 subsets of only $\{3\ell, 4\ell, 6\ell\}$.

$$Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} \left(\frac{5}{4}\right)^{n/24} \left(\frac{8}{6}\right)^{n/24} = 2^{5n/24} 3^{n/8} 5^{n/24} \approx 1.5456^n$$

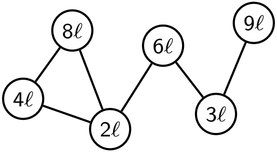
# Constructing primitive sets: One more wrinkle

Consider $k \in \left(\frac{n}{5}, \frac{n}{4}\right]$, the next interval.

If $k = 2\ell$ is even we must consider $2\ell$, $4\ell$, $6\ell, 8\ell$, as well as $3\ell$.




However, if $k \leq \frac{2n}{9}$ then $9\ell \leq n$ must also be considered.




So now we consider $2\ell, 3\ell, 4\ell, 6\ell, 8\ell$ and $9\ell$.

# Computing $Q(n)$

Let $r(k,n) = \dfrac{\#\text{Primitive subsets of [k,n]}}{\#\text{Primitive subsets of [k+1,n]}}$.

Observations:

- $1 \le r(k,n) \le 2$
- $r(k,n)$ only depends on the structure of the connected component of $k$ in the $\mathcal{D}_{[k,n]}$.
- $Q(n) = \displaystyle\prod_{k=1}^{n} r(k,n)$

# Goal: group together like terms.

Fix $k \le n$. Let $i = \left\lfloor \frac{n}{k} \right\rfloor$. The multiples of $k$ up to $n$ are $k$, $2k$, $\ldots ik$.

If one integer in $[k, n]$ divides another the ratio must be $i$-smooth.

Write $k = \ell d$ where $d = \max\{e | k : P^+(e) \le i\}$ is the largest $i$-smooth divisor of $k$ and $\ell = \frac{k}{d}$ is "rough" part of $k$.

Every integer in the connected component of $k$ is divisible by $\ell$.

Let $t = \left\lfloor \frac{n}{\ell} \right\rfloor$, so $t\ell$ is the largest integer up to $n$ divisible by $\ell$.

Then $r(k, n) = r(d\ell, t\ell) = r(d, t)$.

As $n \to \infty$, the number of $k \le n$ sharing the same "$d$" and "$t$" is

$$\frac{n(1 + o(1))}{t(t+1)} \prod_{p<i} \left(1 - \frac{1}{p}\right).$$

# Approximating $Q(n)$

$$Q(n) = \prod_{k=1}^{n} r(k,n) = \prod_{i=1}^{n} \prod_{\substack{d \\ P^+(d)<i}} \prod_{t=id}^{(i+1)d-1} r(d,t)^{\frac{n(1+o(1))}{t(t+1)} \prod_{p<i} \frac{p-1}{p}}.$$

Setting $c = \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d)<i}} \prod_{t=id}^{(i+1)d-1} r(d,t)^{\frac{1}{t(t+1)} \prod_{p<i} \frac{p-1}{p}}$ we get $Q(n) = c^{n+o(n)}$.

## Theorem

*For any $\epsilon > 0$, we have $Q(n) = c^{n\left(1+O\left(\exp\left((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)\right)}$.*

*The constant $c$ is effectively computable and $1.5729 < c < 1.5745$.*

# Numerical Bound

Get a lower bound for $c$ by computing terms in this product. Using Sage to compute $r(d,t)$ for all $i,d,t$ with $di^5 \leq 10^8$ gives $1.5729 < c$.

This technique isn't as good computing upper bounds. Get $c < 1.58$.

The bound $c < 1.5745$ was obtained by Liu, Pach and Palincza (2018) using different methods.

# A general theorem

## Theorem

*Suppose $f(k,n)$ depends only on the connected component of $k$ in $\mathcal{D}_{[k,n]}$ and is bounded $|f(k,n)| \le A$. Then for any $\epsilon>0$,*

$$\sum_{k=1}^{n} f(k,n) = nC_f + O\left(An\exp\left((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)$$

*where*

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d)<i}} \sum_{t=id}^{(i+1)d-1} \frac{f(d,t)}{t(t+1)} \prod_{p<i}\left(1-\tfrac{1}{p}\right).$$

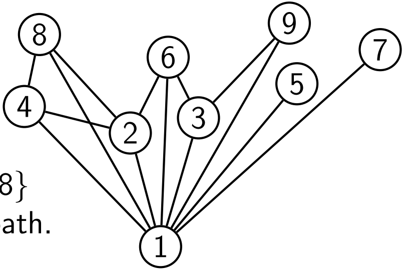
# Path covers of the divisor graph

Pomerance, Erdős, Saias and others study the the length of the longest path in the divisor graph of $[1, n]$, showing it is $\asymp \frac{n}{\log n}$.

Let $C(n)$ be the least number of disjoint paths that contain all the vertices of this graph.

**Example:** $C(9) = 2$

The divisor graph $\mathcal{D}_{[1,9]}$ can be covered by $\{7, 1, 5\}$ and $\{9, 3, 6, 2, 4, 8\}$ but it is not possible to use a single path.



# Path covers of the divisor graph

Bounds on the path cover number $C(n)$ of $\mathcal{D}_{[1,n]}$ have improved.

**Saias** (2003): $\frac{n}{6} \leq C(n) \leq \frac{n}{4}$ for sufficiently large $n$.
**Mazet** (2006): $C(n) \sim \nu n$ for some $0.1706 \leq \nu \leq 0.2289$.
**Chadozeau** (2008): $C(n) = \nu n \left(1 + O\left(\frac{1}{\log\log n \log\log\log n}\right)\right)$.

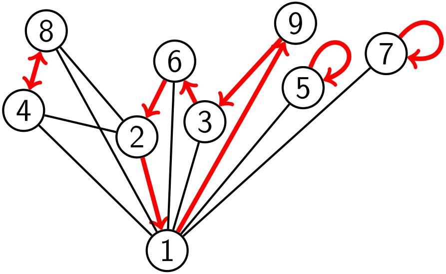
Set $f(k,n) = \#\{\text{paths to cover } \mathcal{D}_{[k,n]}\} - \#\{\text{paths to cover } \mathcal{D}_{[k+1,n]}\}$.
$f(k,n) \in \{-1,0,1\}$ and $C(n) = \sum_{k=1}^{n} f(k,n)$.

**Theorem**

$C(n) = \nu n \left(1 + O\left(\frac{1}{e^{(1-\epsilon)\sqrt{\log n \log\log n}}}\right)\right)$ *and* $0.1909 < \nu < 0.2179$.

# Back to permutations

$$\left\{\begin{matrix}\text{Permutations } \sigma \text{ of } n \text{ where} \\ \forall\, i \text{ either } \sigma(i) \mid i \text{ or } i \mid \sigma(i)\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Directed, vertex-disjoint cycle} \\ \text{covers of } \mathcal{D}_{[1,n]} \text{ with self-loops}\end{matrix}\right\}$$




$$\sigma = 916852743 = (1\,9\,3\,6\,2)(4\,8)(5)(7)$$

Let $d(k,n) = \dfrac{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k,n]}}{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k+1,n]}}$, $D(n) = \prod_{k=1}^{n} d(k,n)$.

# $d(k,n)$

$$d(k,n) = \frac{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k,n]}}{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k+1,n]}}, \quad D(n) = \prod_{k=1}^{n} d(k,n).$$

Seems like the right setup to apply the general theorem....

...Except it isn't clear that $d(k,n)$ is bounded.

In fact it is unbounded, $d(k,n) > \pi\left(\frac{n}{k}\right) - \pi\left(\frac{n}{2k}\right)$.

# $d(k, n)$

It isn't too difficult to adapt the proof of the general theorem to handle a function $f(k, n)$ that doesn't grow too quickly with $\frac{n}{k}$.

Doesn't seem trivial to find any sub-exponential bound for $d(k, n)$.

For any graph $G$ let $D(G)$ denote the count of directed vertex-disjoint cycle covers of $G$ (allowing loops at each vertex.) $D(n) = D(\mathcal{D}_{[1,n]})$.

Let $R(G, v) = \dfrac{D(G)}{D(G \setminus \{v\})}$.

**Theorem**

*For any graph $G$ and vertex $v$ of degree $d$, (counting a self loop)*

$$R(G, v) \leq 1 + \frac{d^2 - d}{2}.$$

# Proof outline

## Theorem

*For any graph $G$ and vertex $v$ of degree $d$, (counting a self loop)*

$$R(G,v) = \frac{D(G)}{D(G \setminus \{v\})} \leq 1 + \frac{d^2 - d}{2}.$$

Let $W = \{w_1, w_2, \ldots, w_{d-1}\}$ be the neighbors of $v$ in $G$.

Write $D(G) = C_v + \sum_{i=1}^{d-1} C_{vw_i} + 2 \sum_{1 \leq i < j < d} C_{w_i v w_j}$.

- $C_v = D(G \setminus \{v\})$ counts cycle covers where $v$ is fixed (a 1-cycle).
- $C_{vw_i}$ counts cycle covers where $v$ is in a 2-cycle with $w_i$.
- $C_{w_i v w_j}$ counts cycle covers containing the edges $w_i \to v \to w_j$.

$$R(G,v) = \frac{D(G)}{C_v} \leq 1 + (d-1) + 2 \sum_{1 \leq i < j < d} \frac{C_{w_i v w_j}}{C_v}$$

# Proof outline

$$R(G,v) = \frac{D(G)}{C_v} \leq 1 + (d-1) + 2\sum_{1\leq i<j<d} \frac{C_{w_i v w_j}}{C_v}$$

**Claim:** $\left(C_{w_i v w_j}\right)^2 \leq \frac{1}{4}\left(C_v\right)^2.$

Write $C_v = X + Y_i + Y_j + Z$

- $X$ counts the cases where $v$, $w_i$, and $w_j$ are all part of 1-cycles
- $Y_i$ counts those where only $v$ and $w_i$ are fixed (not $w_j$).
- $Z$ counts those where $v$ is fixed but neither $w_i$ nor $w_j$ is fixed.

Then write

$$(C_v)^2 = (X+Y_i+Y_j+Z)^2 \geq (X+Z)^2 + (Y_i+Y_j)^2 \geq 4(XZ + Y_iY_j)$$

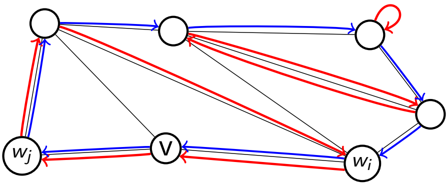
by the AM-GM inequality. It remains to show $XZ + Y_iY_j \geq (C_{w_i v w_j})^2$.

# $XZ + Y_i Y_j \geq (C_{w_i v w_j})^2$

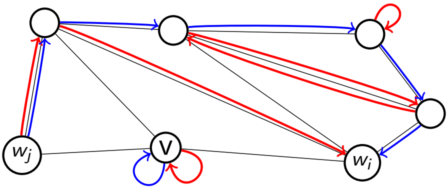
Find an injection from objects counted by $(C_{w_i v w_j})^2$ to objects counted by $XZ + Y_i Y_j$.

Each of $XZ$, $Y_i Y_j$ and $(C_{w_i v w_j})^2$ count pairs of directed vertex-disjoint cycle covers. In each pair, color the first one blue, and the second red.

Draw both on the same graph. Get a colored, directed multigraph, every vertex has one inward and outward pointing edge of each color.



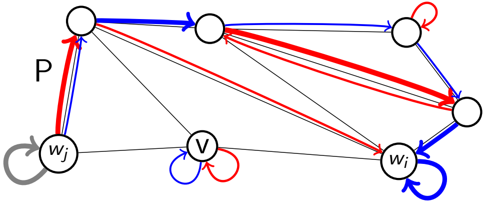
# Colorings!




Take a colored multigraph obtained from $(C_{w_i v w_j})^2$. (It has both blue and red edges $w_i \to v \to w_j$.) Remove all four of these edges.

Add two loops to $v$ one of each color.

Now every vertex (except $w_i$, $w_j$) has in- and out-edges of each color. $w_i$ has in-edges of each color, and $w_j$ has out-edges of each color.



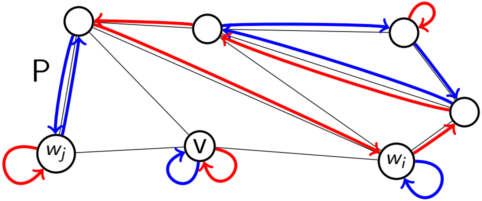

The colored multigraph consists of alternating-color cycles, plus two alternating-color paths $w_j \to w_i$. Call the path ending in a blue edge $P$.

Now add a (initially uncolored) loop to each of the vertices $w_i$ and $w_j$.

Color the new loop on $w_i$ blue. Recolor and reverse every edge along $P$. Every vertex along $P$ (except $w_j$) has a consistent coloring.

Both edges adjacent to $w_j$ have the same color and opposite orientations. Color the new loop at $w_j$ the opposite color.

# Wrapping up




The final result is a coloring in which $v$ is fixed by both colors, every vertex has exactly one in-edge and out-edge of each color.

If both the self-loops on $w_i$ and $w_j$ have the same color (blue) the result is counted by $XZ$, otherwise it is counted by $Y_i Y_J$.

We have shown an injection from objects counted by $(C_{w_i v w_j})^2$ to objects counted by $XZ + Y_i Y_j$. So $(C_{w_i v w_j})^2 \leq XZ + Y_i Y_j \leq \frac{1}{4}(C_v)^2$

# Wrapping up

$$R(G,v) = 1 + (d-1) + 2\sum_{1\le i<j<d} \frac{C_{w_i v w_j}}{C_v}$$
$$= 1 + (d-1) + \frac{(d-1)(d-2)}{2} = 1 + \frac{d^2-d}{2}$$

Using this, we find that

$$d(k,n) = \frac{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k,n]}}{\#\text{ vertex-disjoint cycle covers of } \mathcal{D}_{[k+1,n]}} \le 1 + \frac{\lfloor \frac{n}{k} \rfloor^2 - \lfloor \frac{n}{k} \rfloor}{2}.$$

## Theorem

*For any $\epsilon > 0$, we have $D(n) = c_d^{n\left(1+O\left(\exp\left((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)\right)}$.*

Using $d(k,n) \le 1+\frac{\lfloor \frac{n}{k} \rfloor^2 - \lfloor \frac{n}{k} \rfloor}{2}$ we improve $c_d < 13.6$ to $c_d < 3.32$.

Using numerical computation we improve this to $2.069 < c_d < 2.694$.

# Open questions

What is the best possible constant $C$, $R(G, v) \leq (C + o(1))d^2$ as $d \to \infty$?

We have $C \leq \frac{1}{2}$.

On the other hand, looking at complete bipartite graphs $K_{\frac{n}{2},n}$ we find

$$R(K_{\frac{d}{2},d}, v) \geq \left(\frac{1}{4} + o(1)\right) d^2.$$

# Open questions

What is the length $l(n)$ of the longest path in $\mathcal{D}_{[1,n]}$?

Saias: $3\frac{x}{\log x} \leq l(x) \leq 7\frac{x}{\log x}$, for sufficiently large $x$.

What is the length of the longest cycle? Its at most $l(x)\ldots$

Let L be a path of length $l(n)$ in $\mathcal{D}_{[1,n]}$. What is the length of the longest path in $\mathcal{D}_{[1,n]} \setminus L$? It is $\gg \frac{\sqrt{n}}{\log n}$.

# Other Applications: Maximum Primitive Subsets

Recall that the largest primitive subset of this interval has size $\left\lceil \frac{n}{2} \right\rceil$.

Let $M(n)$ count the primitive subsets of $\{1, \ldots n\}$ of size $\left\lceil \frac{n}{2} \right\rceil$.

**Theorem (Vijay, 2018)**

$$1.141^n < M(n) < 1.187^n$$

**Theorem (Liu, Pach, Palincza, 2018, M., 2018)**

$$M(n) = \alpha^{n\left(1+O\left(\exp\left((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)\right)}, \; 1.14819 < \alpha < 1.14823.$$

# Other Applications: Maximal Primitive Sets

A primitive subset of $[1, n]$ is maximal if it is not contained in another primitive subset. Let $m(n)$ count maximal primitive subsets of $[1, n]$.

## Theorem

$m(n) = \beta^{n\left(1+O\left(\exp\left((-1+\epsilon)\sqrt{\log n \log\log n}\right)\right)\right)}$ *and* $1.2125 < \beta < 1.2409$.

## Corollary

*Let* $V(n)$ *denote the median size of primitive subsets of* $[1, n]$. *Then*

$$0.1681n < V(n) < 0.3918n$$

## Question

*Is* $V(n) \sim vn$ *for some* $v$*? If so, is* $v$ *computable?*

THANK YOU!