

COUNTING PRIMITIVE SUBSETS AND OTHER STATISTICS OF THE DIVISOR GRAPH OF $\{1, 2, \dots, n\}$

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ABSTRACT. Let $Q(n)$ denote the count of the primitive subsets of the integers $\{1, 2, \dots, n\}$. We give a new proof that $Q(n) = \alpha^{(1+o(1))n}$ which allows us to give a good error term and to improve upon the lower bound for the value of this constant α . We also show that the method developed can be applied to many similar problems that can be stated in terms of the divisor graph, including other questions about primitive sets, geometric-progression-free sets, and the divisor graph path-cover problem.

1. PREVIOUS WORK ON COUNTING PRIMITIVE SETS

A set of integers is called a *primitive set* if no integer in the set is a divisor of another. For example the prime numbers form a primitive set, as do the integers with exactly k prime factors for any fixed k .

In 1990, as part of a paper [3] filled with conjectures and problems related to subsets of the integers with various properties, Cameron and Erdős considered the counting function $Q(n)$ of the number of primitive subsets of the integers up to n . They note that the bounds $(\sqrt{2})^n < Q(n) < 2^n$ can be observed immediately from the observation that for fixed n , the set of integers $\{\lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2, \dots, n\}$ is primitive (a set of size $\lceil \frac{n}{2} \rceil$), as are any of the $2^{\lceil \frac{n}{2} \rceil}$ subsets of this set.

Based on this observation, they conjecture that $\lim_{n \rightarrow \infty} Q(n)^{1/n}$ exists. In addition to this conjecture, they outline a proof that the bounds above can be improved to

$$1.55967^n \ll Q(n) \ll 1.60^n. \tag{1}$$

Many of the conjectures in [3] have attracted a substantial amount of attention, especially those of an additive nature. More recently some of the multiplicative questions have begun to attract attention. Recently, their conjecture above about the count of primitive sets was proven by Angelo [1].

Theorem 1 (Angelo, 2017). *The limit $\lim_{n \rightarrow \infty} Q(n)^{1/n}$ exists. Equivalently, there exists a constant α such that $Q(n) = \alpha^{(1+o(1))n}$.*

The proof proceeds by considering subsets of the integers $[1, n]$ in which the ratio between any two included elements is not an (s -smooth) integer (an integer without any prime factors larger than s) for fixed s , and then allowing s to tend to infinity. Unfortunately the proof was not effective in the sense that it doesn't give a way to improve the bounds in (1).

More recently Vijay [19] considered the problem of counting primitive subsets of $[1, n]$ of maximum size, called *maximum primitive sets*. A pidgeonhole principle argument shows that for any n the maximum size of a primitive subset of the integers up to n is $\lceil \frac{n}{2} \rceil$. He considers only even n , although the result extends

immediately to all integers n . In particular he shows that the count $M(n)$ of the number of primitive subsets of $[1, n]$ of size $\lceil \frac{n}{2} \rceil$ satisfies the bounds

$$1.303^n \ll M(2n) \ll 1.408^n$$

or, in terms of n ,

$$1.141^n \ll M(n) \ll 1.187^n.$$

These questions have also recently been investigated independently by Liu, Pach and Palincza [8], and some of the results of that paper overlap with the results presented here. While many of the ideas are similar, we take a slightly different approach that gives an error term and computationally seems to provide better lower bounds, though is not so good at giving numerical upper bounds. They present a method to effectively compute α , and improve the bounds in (1) to

$$1.571068^n \ll Q(n) \ll 1.574445^n.$$

They also consider maximal primitive sets, and show that the limit $\lim_{n \rightarrow \infty} M(n)^{1/n}$ exists, (which we denote β) show that this constant can be effectively computed, and improve the bounds above to

$$1.148172^n \ll M(n) \ll 1.148230^n \quad \text{or} \quad 1.3183^n \ll M(2n) \ll 1.31843^n.$$

2. RESULTS AND OTHER APPLICATIONS

We show that these two constants α and β can be given explicitly in terms of an infinite product described in Section 3. This observation allows us to prove the following.

Theorem 2. *For any $\epsilon > 0$, and n sufficiently large, the number of primitive subsets of $[1, n]$ is*

$$Q(n) = \alpha^n \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right).$$

or equivalently,

$$\log Q(n) = n \log \alpha + O \left(n \exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right)$$

and the constant α is effectively computable.

This result is a corollary of our main theorem, which can be used to compute a wide variety of statistics about subsets of $\{1, 2, \dots, n\}$ with certain multiplicative structure. Before we can state our main theorem however we must introduce a definition. Consider the divisor graph of a set of integers, the graph obtained by treating each number as a vertex and connecting two integers by an edge if one divides another. In this context, for example, a primitive subset would correspond to an independent set of vertices.

We also consider the divisor graphs of the integers in an interval $[a, n]$. In general these graphs won't be connected, and we will be interested primarily in the component of this graph connected to the vertex a . We say that a function $f(a, n)$ depends only on the connected component of a in the divisor graph of the interval $[a, n]$ if $f(a, n) = f(b, m)$ whenever the connected component of b in the divisor graph of $[b, m]$ is isomorphic to that of a in $[a, n]$ (with the vertex of a corresponding to b in the isomorphism). With this, we can state our main theorem.

Theorem 3 (Main Theorem). *Suppose $\epsilon > 0$, $A \geq 0$ and $f(a, n)$ is a bounded function $|f(a, n)| \leq A$ that depends only on the connected component of a in the divisor graph of the interval $[a, n]$. Then there exists a constant*

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id, (i+1)d)} \left(\frac{f(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right)$$

such that

$$\sum_{a=1}^n f(a, n) = nC_f + O_A \left(n \exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right).$$

As we will see, many questions about subsets with multiplicative structure can be regarded as questions about this divisor graph, and we demonstrate several different applications of this result. First we apply it to several related problems about primitive sets. For example, we obtain the corresponding result for counting maximum primitive subsets as well.

Theorem 4. *The limit $\lim_{n \rightarrow \infty} M(n)^{1/n} = \beta$ exists. Furthermore, for any $\epsilon > 0$ and n sufficiently large,*

$$M(n) = \beta^n \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right)$$

and the constant β is effectively computable.

We also use it to count maximal primitive subsets of $[1, n]$, where a primitive subset is maximal if no additional integer from the interval can be added to the set without having one integer divide another in the subset. Note that all maximum primitive subsets are maximal, but not all maximal primitive subsets are a maximum primitive set. For example, $\{2, 3, 5, 7\}$ is a maximal primitive subset of the integers up to 9, but not a maximum primitive set, as $\{5, 6, 7, 8, 9\}$ is larger. We obtain analogues of these theorems for $m(n)$, the count of the maximal primitive subsets of $[1, n]$.

Theorem 5. *The limit $\lim_{n \rightarrow \infty} m(n)^{1/n} = \eta$ exists. Furthermore, for any $\epsilon > 0$ and n sufficiently large,*

$$m(n) = \eta^n \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right)$$

and the constant η is effectively computable.

By estimating the relevant constant from Theorem 3 we can numerically approximate each of these constants to arbitrary precision. In Section 4 we describe the computations performed to compute partial sums of this product, which give us bounds for the size of these constants.

Theorem 6. *The constants α , β , and η satisfy the bounds*

$$1.572939 < \alpha$$

$$1.148192 < \beta \quad (\text{or } 1.318345 < \beta^2)$$

$$1.212500 < \eta < 1.240904.$$

The lower bounds for α and β are stronger than those appearing in [8] however our method has not yet been able to improve upon the upper bounds obtained by their methods. Combining their result with ours gives the new known ranges

$$1.572939 < \alpha < 1.574445.$$

$$1.148192 < \beta < 1.148230 \quad (\text{or } 1.318345 < \beta^2 < 1.31843).$$

The constant η and the problem of counting maximal primitive subsets of the integers does not appear to have been considered before in the literature, so in this situation we have included the upper bound obtained by our method as well.

Despite our method's success at counting primitive subsets with a variety of properties, it isn't clear that it can be used to count primitive subsets of sizes other than $\lceil \frac{n}{2} \rceil$. It would be interesting, for example, to study the distribution of the sizes of the primitive subsets of $\{1, 2, \dots, n\}$. As a partial result in this direction we use the bounds for α and η from Theorem 6 to bound the median size of such a subset.

Theorem 7. *Let $\nu(n)$ denote the median of the sizes of the primitive subsets of $\{1, 2, \dots, n\}$. Then for sufficiently large n , we have*

$$0.168153n < \nu(n) < 0.391752n.$$

It is natural to conjecture from this that $\nu(n) \sim vn$ for some constant v , this and other questions are posed in Section 9.

We briefly introduce a few additional applications of the main theorem to problems outside of primitive sets. In Section 7 we consider geometric progression free subsets of the integers up to n . Here we consider geometric progressions of the form a, ar, ar^2 with a a positive integer, and r an integer ratio greater than 1. Unlike the case of primitive sets, where it is easy to see that the largest primitive subset of $\{1, 2, \dots, n\}$ has size $\lceil \frac{n}{2} \rceil$, the problem of determining the largest subset of these integers avoiding 3-term geometric progressions is not so clear. Various authors have considered the problem of determining the greatest possible density of a set of integers that is free of 3-term geometric progressions. (See [13], [14], [2], [11], [10]). Let $G(n)$ denote the size of the largest subset of the integers up to n avoiding a 3-term geometric progression with integral ratio. The argument for the bounds for the upper density of a subset of integers avoiding such progressions in [10] shows that

$$0.81841n \leq G(n) \leq 0.81922n$$

and that $G(n) \sim bn$ for an effectively computable constant b . The main theorem allows us to improve this result with an error term.

Theorem 8. *For any $\epsilon > 0$, $G(n)$ the size of the largest geometric progression free subset of the integers up to n satisfies*

$$G(n) = bn \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right).$$

We can likewise count the geometric progression free subsets of these integers, which we denote by $H(n)$.

Theorem 9. *The limit $\lim_{n \rightarrow \infty} H(n)^{1/n} = \theta$ exists. Furthermore, for any $\epsilon > 0$ and n sufficiently large,*

$$H(n) = \theta^n \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right).$$

The constant θ is effectively computable and satisfies the bounds

$$1.901448 < \theta < 1.925556.$$

In Section 8 we consider the problem of covering the divisor graph of $\{1, 2, \dots, n\}$ with as few vertex-disjoint paths as possible. (A minimal path cover of the graph) Various authors have studied the paths in this divisor graph. Pomerance [12] showed that the length of the longest vertex-disjoint path in this divisor graph has length $o(n)$ and now, due to Erdős and Saias [5], we know that the length of this longest path can be bounded above and below by positive constants multiplied by $\frac{n}{\log n}$.

Let $C(n)$ denote the minimal number of vertex disjoint paths required to cover the integers up to n . For example $C(7) = 2$, as the divisor graph can be covered by the two paths $\{7, 1, 5\}$ and $\{3, 6, 2, 4\}$ but it is not possible to include all the vertices in a single path. This problem has been considered in several papers after being introduced by Saias and Erdős [5]. Saias [15] showed that $\frac{n}{6} \leq C(n) \leq \frac{n}{4}$ for sufficiently large n . Mazet [9] improves this, showing that $C(n) \sim cn$ for some constant c satisfying $0.1706 \leq c \leq 0.2289$, and Chadozeau [4] gives the error term $C(n) = cn \left(1 + O \left(\frac{1}{\log \log n \log \log \log n} \right) \right)$. Our main theorem allows us to improve this error term dramatically and also to improve the lower bound for c .

Theorem 10. *The minimum number of paths required to cover the divisor graph of the integers up to n satisfies*

$$C(n) = cn \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right).$$

The constant c is effectively computable, and satisfies the improved lower bound $0.176488 < c$.

Finally, in Section 10 we conclude with several lemmas from analytic number theory which lead to the proof of the main theorem.

Smooth numbers play an important role throughout the paper. We will denote by $P^+(n)$ the largest prime divisor of n (and take the convention that $P^+(1) = 1$), and by $P^-(n)$ the smallest prime divisor of n .

3. COUNTING PRIMITIVE SETS

We first show how the main theorem, Theorem 3, can be used to estimate $Q(n)$, the count of the primitive subsets of the integers up to n . We fix n , and count the possible primitive subsets of $\{1, 2, \dots, n\}$ by working backward from the end. Recall, any subset of the integers in $\left(\frac{n}{2}, n\right]$ is primitive. Since each element of this set can either be included or not so $Q(n) \geq 2^{n/2}$. We can make this bigger by considering integers those integers in $\left(\frac{n}{3}, \frac{n}{2}\right]$. For each such k in this range we can either include $k, 2k$ or neither. Thus for each such integer there are 3 possibilities, replacing the two possibilities when only $2k$ was considered. Thus $Q(n) \geq 2^{n/2} \left(\frac{3}{2}\right)^{n/6} = 2^{n/3} 3^{n/6} \approx 1.5131^n$.

We generalize this to any $k \leq n$ by defining

$$r(k, n) = \frac{\#\text{Primitive subsets of } [k, n]}{\#\text{Primitive subsets of } [k+1, n]} \quad (2)$$

to be the contribution of k to our potential primitive set working backward from the end.

With this definition we can see that

$$Q(n) = \prod_{k=1}^n r(k, n)$$

as this product telescopes, leaving only $Q(n) = \frac{\#\text{Primitive subsets of } [1, n]}{\#\text{Primitive subsets of } \emptyset}$.

We can apply Theorem 3 by taking $f(k, n) = \log(r(k, n))$. To do so, it is necessary to see that $f(i, n)$ depends only on the connected component of k in the divisibility graph of $[k, n]$. This is easily seen however, as the inclusion of k (or any integer in the interval $[k, n]$) in a primitive set depends only on whether its multiples (or divisors) are already included. The count of the number of primitive subsets of an interval can therefore be computed as the product of the number of primitive subsets (or independent sets of vertices when viewed as a graph) of each connected component of the divisibility graph. The contribution from each component of the divisibility graph that doesn't contain k will cancel in the numerator and denominator of (2). This leaves only the ratio of the number of primitive (independent) subsets of the component connected component of k to the number of primitive (independent) subsets of this component with the element k removed. As this is purely a graph theoretic question, it is clear that the answer will be the same for any other isomorphic divisibility graph.

Note also that $1 \leq r(k, n) < 2$, since every primitive subset of $[k+1, n]$ is also a primitive subset of $[k, n]$, and at most the number of primitive sets could double after k is considered if it is possible to add k to every primitive subset of $[k+1, n]$ and still get a primitive set. Thus we have $0 \leq f(k, n) \leq \log 2$.

Applying Theorem 3 we have

$$\begin{aligned} Q(n) &= \prod_{i=1}^n r(i, n) = \exp\left(\sum_{i=1}^n f(i, n)\right) \\ &= \exp\left(n \log \alpha + O\left(n \exp\left(-\sqrt{\left(\frac{1}{6}-\epsilon\right) \log n \log \log n}\right)\right)\right) \end{aligned}$$

with

$$\begin{aligned} \alpha &= \exp\left(\sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id, (i+1)d]} \left(\frac{f(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p}\right)\right) \\ &= \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d) \leq i}} \prod_{t \in [id, (i+1)d]} \left(\frac{r(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p}\right). \end{aligned} \quad (3)$$

This proves Theorem 2.

The proofs of Theorems 4 and 5 are very similar. We define

$$s(k, n) = \frac{\#\text{Primitive subsets of } [k, n] \text{ of maximum size}}{\#\text{Primitive subsets of } [k+1, n] \text{ of maximum size}} \quad (4)$$

and

$$w(k, n) = \frac{\#\text{Maximal primitive subsets of } [k, n]}{\#\text{Maximal primitive subsets of } [k+1, n]}. \quad (5)$$

It follows that these functions depend only on the connected component of k in the divisibility graph of $[k, n]$ by the same argument as for $r(k, n)$. Unlike the case of $r(k, n)$, however, some care is required to show that these functions are bounded so that Theorem 3 can be applied.

First note that if $k > \frac{n}{2}$ then $s(k, n) = w(k, n) = 1$, since in this situation, k is the only vertex in the component connected to k in the divisor graph of $[k, n]$, and so the element k must be included in a maximal/maximum primitive subset of $[k, n]$. Thus we can restrict our attention to the case $k \leq \frac{n}{2}$.

For either $s(k, n)$ or $w(k, n)$, we find that every maximum/maximal primitive subset of $[k+1, n]$ is also a maximum/maximal primitive subset of $[k, n]$ of maximal size, since one of the multiples of $2k$ (and thus of k) must have been included in the primitive set if it were to be maximal. Thus both $s(k, n)$ and $w(k, n)$ are always at least 1.

To get upper bounds for these functions, we produce an injection from maximal/maximum primitive subsets of $[k, n]$ containing k to maximal/maximum primitive subsets of $[k+1, n]$. First note that any maximum primitive subset of $[k, n]$ that contains k can be put into correspondence with a primitive subset of $[k+1, n]$ of the same size by replacing k by $2k$. (Note that every neighbor of $2k$ in the divisor graph is also a neighbor of k). This is clearly injective, so $s(k, n) \leq 2$. For maximal primitive sets, simply replacing k by $2k$ might not produce a maximal primitive subset if it is possible also to add other (odd) multiples of k once k is removed. We therefore remove k and instead include pk for each prime number p with $pk \leq n$ that can be added to the set while still producing a primitive set. Since it is always possible to add the element $2k$ once k is removed, it is clear that this again is an injection and so $w(k, n) \leq 2$ as well.

We then set

$$M(n) = \prod_{k=1}^n s(k, n), \quad m(n) = \prod_{k=1}^n w(k, n)$$

which each telescope in the same way as the product for $Q(n)$, and apply Theorem 3, proving Theorems 4 and 5 with

$$\beta = \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d) \leq i}} \prod_{t \in [id, (i+1)d]} \left(\frac{s(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right) \quad (6)$$

and

$$\eta = \prod_{i=1}^{\infty} \prod_{\substack{d \\ P^+(d) \leq i}} \prod_{t \in [id, (i+1)d]} \left(\frac{w(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right). \quad (7)$$

4. NUMERICAL ESTIMATES

Using the product formulas for α , β and η , given in (3), (6) and (7) respectively, it is possible to compute each constant to arbitrary precision by computing sufficiently many terms in the product. Replacing any unestimated terms by an appropriate upper or lower bound for the term gives a corresponding upper or lower bound for the value of the constant.

The constants were estimated using computations done using Sagemath [18]. Several optimizations were made to improve the speed of convergence.

Observation 1. *For a fixed value of i and d , if $t + 1 \in (id, (i + 1)d)$ is not an i -smooth number then $r(d, t) = r(d, t + 1)$.*

This follows because the connected component of d in the divisibility graph of the interval $[d, t + 1]$ does not include the integer $t + 1$ in that case. Thus for each pair i, d it is only necessary to compute $r(d, t)$ for the i -smooth values of t in the interval $[id, (i + 1)d)$.

Observation 2. *Fix i, d and $t \in [id, (i + 1)d)$. If, for some $p \leq i$, every term in the connected component of pd in the divisibility graph of $[pd, pt]$ is divisible by p , then $r(pd, pt) = r(d, t)$. Furthermore, in this situation $r(p^j d, p^j t) = r(d, t)$ for all $j > 0$.*

Proof. If every term in the connected component of pd in the divisibility graph of this interval is divisible by p , then it is clear that this component is isomorphic to the component obtained by dividing everything by p .

Now, suppose every term in this component is divisible by p , and consider the interval $[p^2 d, p^2 t]$. Clearly the component connected to $p^2 d$ contains every term in the component of the divisibility graph of $[d, t]$ connected to d multiplied by p^2 . Now suppose for contradiction that the connected component of dp^2 contained an additional term $a > dp^2$ not divisible by p^2 . Then this term must be a divisor of some term bp^2 in the connected component of dp^2 where $p \nmid b$ (or if not, it could be replaced by one that is, since there must exist a path in this graph connecting a to dp^2).

This means the ratio between a and bp^2 is divisible by p , and so

$$bp^2/p = bp \geq a > dp^2.$$

Thus we have $b > dp$ and so b would be an element of the connected component of dp in the divisor graph of $[pd, pt]$ which is a contradiction. \square

Observation 2 means that the sum (or product) over d becomes finite for values of $i \leq 4$, because values of $d > 48$ will always result in a connected graph component with a common prime factor in this range. Thus the challenge is to numerically estimate the terms for $i \geq 5$.

Determining the optimal order in which to consider the pairs of i and d in the infinite product is challenging because $r(d, t)$ becomes more difficult to compute both for larger values of d and t . At the same time, the value of $r(d, t)$ tends to one rapidly as $i = \lfloor \frac{t}{d} \rfloor$ gets large. Experimentally, the best lower bounds obtained by the author were obtained by considering potential pairs i, d in increasing order of the value of the expression $d \times i^5$.

By computing $r(d, t)$ For all pairs of i, d with d being an i -smooth integer satisfying $di^5 \leq 10^8$, (as well as the extended ranges $d < 11250000$ when $i = 5$,

$d < 2400000$ for $i = 6$ and $d < 27440$ when $i = 7$) and all i -smooth values of $t \in [id, (i+1)d)$, and using these terms in the product (3) along with the additional terms obtained by applying Observation 2, the lower bound is obtained.

In a similar fashion one can compute initial terms for β and η . The computation for β involved computing $s(d, t)$ for all values of d and i satisfying $di^5 < 10^8$. For η a more involved range of values was taken into account in order to get decent upper and lower bounds. Table 1 gives the bounds for the d taken into account when computing η . Again, in both computations Observations 1 and 2 were used to speed up the computation and take into account additional pairs d, t .

5. CONJECTURAL IMPROVED UPPER BOUNDS

We give here a different method of counting primitive sets which would give a much better upper bound for the constant α if Conjecture 1 were proven, and may be of some independent interest.

Here we count primitive subsets of $\{1, 2, \dots, n\}$ by working forward from 1, rather than backward from n as was done in the previous section. We define $g(1) = 1$ and for $k > 1$

$$g(k) = \frac{Q(k)}{Q(k-1)} - 1 = \frac{\#\text{Primitive subsets of } [1, k] \text{ that include } k}{\#\text{Primitive subsets of } [1, k-1]}.$$

The reason we defined $g(k)$ this way, as the ratio of primitive sets up to k that include the integer k to the primitive sets up to k that do not, is that the resulting function appears to be submultiplicative in the following sense.

Conjecture 1. *The function $g(n)$ is submultiplicative. If $(n, m) = 1$ then*

$$g(nm) \leq g(n)g(m).$$

Furthermore, for any prime p , $g(p^{i+1}) \leq g(p^i)$.

The conjecture has been verified for all values of $n < 899$.

If one assumes the conjecture, then a substantially improved upper bound for α can be computed. Using the computed values of $g(n)$, $n < 899$, and taking into account all 31-smooth numbers gives the upper bound $c < 1.573487$.

6. THE MEDIAN SIZE OF A PRIMITIVE SUBSET

The computations obtained in the previous section will also allow us to give upper and lower bounds for $\nu(n)$, the median size of a primitive subset of the integers up to n . First we need a lemma (see for example [7]) on sums of binomial coefficients, which gives the number of subsets of a sets of size n into subsets of size at most λn .

Lemma 1. *Fix $0 < \lambda < \frac{1}{2}$. The partial sum of binomial coefficients satisfies*

$$\sum_{i=1}^{\lfloor \lambda n \rfloor} \binom{n}{i} = 2^{nH(\lambda) + O(\log n)}$$

where $H(\lambda) = -\lambda \log_2 \lambda + (\lambda - 1) \log_2 (1 - \lambda)$ and \log_2 is the logarithm base 2.

Using this, and the numerical computations of α and β , we can bound the median size of a primitive subset of $\{1, 2, \dots, n\}$.

Proof of Theorem 7. Let $\nu(n)$ denote the median size of the primitive subsets of the integers up to n . Since half of the primitive subsets of this set must have size at most $\nu(n)$, and the total number of primitive subsets is approximately α^n , we can obtain a lower bound for $\nu(n)$ by supposing that all small subsets of the integers up to n are primitive. Combining this with Lemma 1 we get that

$$\sum_{i=1}^{\nu(n)} \binom{n}{i} = 2^{nH(\nu(n)/n)+O(\log n)} \geq \frac{1}{2}Q(n) = \alpha^n \left(1 + O\left(\exp\left(-\sqrt{\left(\frac{1}{4}-\epsilon\right)\log n \log \log n}\right)\right)\right).$$

Taking logs gives

$$H(\nu(n)/n) \geq \frac{\log \alpha}{\log 2} + O\left(\exp\left(-\sqrt{\left(\frac{1}{4}-\epsilon\right)\log n \log \log n}\right)\right)$$

or

$$\nu(n) \geq nH^{-1}\left(\frac{\log \alpha}{\log 2}\right) (1 + o(1)).$$

Using the lower bound $1.572939 < \alpha$ gives $\nu(n) \geq 0.168153n$ for n sufficiently large.

To get an upper bound, we must use in addition the lower bound obtained for the constant η . Every primitive set is contained in a maximal primitive set (but not necessarily a maximum one). We count those primitive sets obtained by starting with a maximal primitive set S and removing some subset T from it. Since half of the primitive subsets up to n have size at least $\nu(n)$, and every maximal primitive set has size at most $\frac{n}{2}$, it must be possible to construct at least half of the primitive subsets up to n by starting with a maximal primitive set and removing at most $\frac{n}{2} - \nu(n)$ terms from it. Thus the median size of a primitive subset must be sufficiently less than $\frac{n}{2}$ to produce $\frac{n}{2}Q(n)$ sets in this way. In particular we have the inequality

$$m(n) \sum_{i=0}^{\frac{n}{2}-\nu(n)} \binom{n/2}{i} \geq \frac{1}{2}Q(n).$$

Using Lemma 1 and Theorems 2 and 5, and taking logs, gives

$$H\left(1 - \frac{2\nu(n)}{n}\right) \geq \frac{2(\log \alpha - \log \eta)}{\log 2} + O\left(\exp\left(-\sqrt{\left(\frac{1}{4}-\epsilon\right)\log n \log \log n}\right)\right)$$

so that $\nu(n) \leq \frac{n}{2} \left(1 - H^{-1}\left(\frac{2(\log \alpha - \log \eta)}{\log 2}\right) + o(1)\right)$. Using the upper bound for α , and the lower bound for η , we find that $\nu(n) \leq 0.391752n$ for sufficiently large n . \square

7. GEOMETRIC PROGRESSION FREE SETS

Geometric progression free sets are similar to primitive sets, and so it is not surprising that the methods used to study primitive sets apply to this situation as well. In this situation the entire divisor graph isn't necessary, only a subgraph that contains 3-term geometric progressions. Unlike primitive sets, we don't know precisely the size $G(n)$ of the largest subset of $\{1, 2, \dots, n\}$ avoiding three term geometric progressions with integral ratio. It is known that $G(n) \sim bn$ for some constant b with $0.81841 < b < 0.81922$, and that b is effectively computable. While

we can't improve these bounds using the current method, we can get in Theorem 8 an improved error term $G(n) = bn \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right)$ for any $\epsilon > 0$.

Proof. In this situation we have a sum, not a product so there is no need to take logs, however the set-up is the same. We let $G(d, t)$ denote the size of the largest geometric progression free subset of the integers $[d, t]$, and let $g(d, t) = G(d, t) - G(d + 1, t)$, so that we can write a telescoping sum,

$$G(n) = G(1, n) = \sum_{k \leq n} g(k, n).$$

The main theorem then applies since $g(d, t)$ depends only on the divisor graph of the interval $[d, t]$, and only ever takes the values 0 or 1. The result follows immediately. \square

We can likewise use the main theorem to count (Theorem 9) the number $H(n)$ of subsets of the integers up to n that avoid 3-term-geometric progressions with integral ratio. The proof is identical to that of Theorem 2. To obtain the bounds on the constant θ , we find that the analogous version of Observation 2 for geometric progressions allows us to take into account the contribution from all values of d for $1 \leq i \leq 11$, as well as the contribution from those values of d up to those listed in Table 2 in order to get the bounds

$$1.901448 < \theta < 1.925556.$$

8. AN OPTIMAL PATH COVERING OF THE DIVISOR GRAPH

As mentioned in the introduction, it is known due to Mazet [9] and Chadozeau [4] that $C(n)$, the minimum number of disjoint paths necessary to cover the divisor graph of the integers up to n satisfies

$$C(n) = cn \left(1 + O \left(\frac{1}{\log \log n \log \log \log n} \right) \right).$$

for a constant $0.1706 \leq c \leq 0.2289$. Using our main theorem, we show the following.

Theorem (Theorem 10). *The minimum number of paths required to cover the divisor graph of the integers up to n satisfies*

$$C(n) = cn \left(1 + O \left(\exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right) \right).$$

The constant c is effectively computable, and satisfies the improved lower bound $0.176448 < c$.

Proof. As in previous examples, we set up a telescoping sum. Define

$$V(k, n) = \#\{\text{disjoint paths needed to cover the divisor graph of } [k, n]\}$$

and define $v(k, n) = V(k, n) - V(k + 1, n)$ so that $C(n) = \sum_{k=1}^n v(k, n)$. Note that $v(k, n) \in \{-1, 0, 1\}$, since in the worst case scenario, we can cover the additional vertex k using a single new path, and in the best case scenario, we are able to join paths from two different components of the divisor graph of $[k + 1, n]$ through the vertex k . Since k can only be part of one path however, it is never possible to decrease the number of paths required by more than 1.

It is also clear that $v(k, n)$ depends only on the connected component of k in the divisor graph of $[k, n]$, so we can apply the main theorem to $C(n)$, which completes the proof, with

$$c = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id, (i+1)d)} \left(\frac{v(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right).$$

Note that $v(1, 1) = 1$, but is otherwise nonpositive for any of the terms included in the sum above. (If $i \geq 2$, then when computing $v(d, t)$, the element $2d$ is a vertex of the graph, and any neighbor of $2d$ is also a neighbor of d . Thus it is always possible to cover the vertex d by including it in the same path that is used to cover $2d$.)

In practice it seems that $v(d, t)$ is generally equal to -1 unless d is divisible by 6 in which case it is generally 0. There are a few exceptions, however, in particular, it was computed that $v(d, t) = -1$ for all values of d, t arising when $i \leq 5$; when $d = 6$ and $t = 50, 51$; when $d = 24$ and $225 \leq t \leq 239$; and when $d = 30$ and either $250 \leq t \leq 269$ or $280 \leq t \leq 299$. The range of d searched for each value of i is listed in Table 3. In addition, a few sporadic values of equal to 0 were found when d was not divisible by 6. These are listed in Table 4. Using these computed values where $v(d, t) = 0$, along with observations 1 and 2, and assuming $v(d, t) = -1$ for all other values, we obtain the improved lower bound $0.176488 < c$. \square

9. QUESTIONS

Despite the wide range of problems that can be tackled using the main theorem, this work leaves several open questions that will require new techniques. In addition to the conjecture described in Section 5, we pose two questions that seem interesting enough to study further.

Question 1. *Is the median size $\nu(n)$ of the primitive subsets of the integers up to n asymptotic to νn for some constant ν ? If so, is there an algorithm to compute ν to arbitrary precision?*

In Section 7 we considered sets that avoided 3-term geometric progressions with integral ratio. The results in [10] imply that the largest subset of the integers up to n avoiding 3-term geometric progressions with *rational* ratio is also asymptotic to a different effectively computable constant times n .

Question 2. *Is it possible to give an error term analogous to Theorem 8 for the size of the largest subset of the integers up to n avoiding 3-term geometric progressions with rational ratio? Can we count the number of such subsets analogously to Theorem 9?*

10. PROOF OF THE MAIN THEOREM

It remains to prove the main theorem upon which the results of this paper rely. First we need a lemma categorizing the connected components of the divisor graph of an interval, as well as several lemmas from analytic number theory.

Lemma 2. *Fix a positive integer n , suppose that $0 < a \leq n$, and set $i = \lfloor \frac{n}{a} \rfloor$. Let $d|a$ be the largest i -smooth divisor of a , and $\ell = \frac{a}{d}$ the “ i -rough” part of a . Finally let $t = \lfloor \frac{n}{\ell} \rfloor$. Then the connected component of a in the divisor graph of $[a, n]$ is*

isomorphic to the connected component of d in the divisor graph of $[d, t]$ (with the vertex of a corresponding to the vertex of d).

Proof. Suppose that $a \leq b < c \leq n$ are any two connected vertices in the connected component of a in this divisor graph, and let $r = \frac{c}{b}$ be the (necessarily integral) ratio between them. Since $r \leq \frac{n}{a} < i + 1$, r cannot have any prime factors greater than i . So b and c are divisible by all of the same prime factors greater than i to the same powers.

Recall that ℓ is divisible only by primes greater than i and $\ell|a$ so ℓ divides all of the integers in the connected component of a . We defined $t = \lfloor \frac{n}{\ell} \rfloor$ so that $t\ell$ is the largest integer less than or equal to n divisible by ℓ , and thus the largest number from this interval that could possibly be part of the component of the divisor graph connected to a . So the connected component of a in $[a, n]$ is the same as the connected component of a in $[a, t\ell]$.

Now we can divide each integer in this connected component by ℓ , and see that the connected component of a in $[a, t\ell]$ is the same as the connected component of $[\frac{a}{\ell}, t] = [d, t]$ with the isomorphism being multiplication by ℓ . \square

Lemma 3. *Let $\epsilon > 0$, I an interval of length $X > y > 2$ and define $u = \frac{\log X}{\log y}$. The number of integers in I free of prime divisors up to y is*

$$X \left(\prod_{p \leq y} \frac{p-1}{p} + O(\log y \exp((-1 + o(1))u \log u)) \right) + O_\epsilon(X^\epsilon)$$

as $u \rightarrow \infty$.

Proof. This follows from the ‘‘Fundamental Lemma’’ of Brun’s sieve, see for example Theorem 6.12 of [6], taking the level of distribution to be X . The error term $R(I, D)$ can be bounded by $\Psi(X, y)$, which means that the number of such integers in this interval is

$$X \prod_{p \leq y} \frac{p-1}{p} (1 + O(\exp((-1 + o(1))u \log u))) + O(\Psi(X, y)).$$

Now we can bound $\Psi(X, y) \ll_\epsilon X u^{-u} + X^\epsilon$ [17, Section III.5] and the result follows by approximating $\prod_{p \leq y} \frac{p-1}{p} = O\left(\frac{1}{\log y}\right)$. \square

We will also need estimates of Tenenbaum [16] for the function $\Theta(x, y, z)$ which counts integers $n \leq x$ with a y -smooth divisor $d > z$, as well as the related function

$$S(y, z) = \sum_{\substack{d > z \\ P^+(d) < y}} \frac{1}{d}.$$

Lemma 4. *For all $\epsilon > 0$, $x, y \geq 2$ and $z < \exp \exp(\log y)^{(3/5-\epsilon)}$*

$$\Theta(x, y, z) = (1 + o(1))xS(y, z) \prod_{p < y} \frac{p-1}{p} \ll \frac{x\rho(u)}{\log(u)}.$$

Proof of main theorem. Fix $\epsilon > 0$, $A \geq 0$ and suppose $f(k, n)$ is bounded in absolute value by A and depends only on the connected component of k in the divisor

graph of $[k, n]$. Our goal is to estimate $\sum_{k=1}^n f(k, n)$ as nC_f where

$$C_f = \sum_{i=1}^{\infty} \sum_{\substack{d \\ P^+(d) \leq i}} \sum_{t \in [id, (i+1)d)} \left(\frac{f(d, t)}{t(t+1)} \prod_{p \leq i} \frac{p-1}{p} \right).$$

We use Lemma 2 to group together the equal terms in this sum. We first group together those k having the same value of $i = \lfloor \frac{n}{k} \rfloor$, followed by those whose largest i -smooth divisor d is the same, and finally those with the same value of $t = \lfloor \frac{nd}{k} \rfloor$. The possible values of t are thus each of the integers in the interval $[id, (i+1)d)$.

Pulling this all together we have

$$\begin{aligned} \sum_{k=1}^n f(k, n) &= f(1, n) + \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\substack{d < \frac{n}{i} \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \sum_{\substack{k \in (\frac{dn}{i+1}, \frac{dn}{i}] \\ d|k \\ P^-(\frac{k}{d}) > i}} f(k, n) \\ &= f(1, n) + \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{\substack{d < \frac{n}{i} \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(f(d, t) \sum_{\substack{\ell \in (\frac{n}{t+1}, \frac{n}{t}] \\ P^-(\ell) > i}} 1 \right). \end{aligned}$$

We truncate the outer two sums at some levels N and M to be determined later, giving us

$$\sum_{k=1}^n f(k, n) = \sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(f(d, t) \sum_{\substack{\ell \in (\frac{n}{t+1}, \frac{n}{t}] \\ P^-(\ell) \geq i}} 1 \right) + O_A \left(\frac{n}{N} + \Theta(n, N, M) \right).$$

In doing so, it was necessary to introduce two error terms, $\frac{n}{N}$ to account for all of those integers $k < \frac{n}{N}$ in the product for $Q(n)$ that were omitted when truncating the initial product at N , and $\Theta(n, N, M)$, which counts those integers up to n which have an N -smooth divisor greater than M , and hence may have been ignored when truncating the second sum.

This means that the intervals being sifted of primes up to i when choosing ℓ have size $\frac{n}{t(t+1)}$ and so, as long as both N and M are chosen not too large (meaning that $t < NM$ is not too large), these intervals are sufficiently long to apply Lemma 3 to approximate the count of such integers. Doing so gives

$$\begin{aligned}
& \sum_{k=1}^n f(k, n) \\
&= \sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(\frac{nf(d, t)}{t(t+1)} \left(\prod_{p < i} \frac{p-1}{p} + O\left(\frac{\log i}{\exp((1+o(1))u_t \log u_t)} \right) \right) \right) \\
&\quad + O_{A, \epsilon} \left(\frac{n^\epsilon}{t^{2\epsilon}} \right) + O_A \left(\frac{n}{N} + \Theta(n, N, M) \right) \\
&= \sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \frac{f(d, t)n}{t(t+1)} \prod_{p < i} \frac{p-1}{p} \\
&\quad + O_A \left(\sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(\frac{n \log i}{t^2 \exp((1+o(1))u_t \log u_t)} + \frac{n^\epsilon}{t^{2\epsilon}} \right) + \frac{n}{N} + \Theta(n, N, M) \right)
\end{aligned} \tag{8}$$

as long as $u_t = \frac{\log \frac{n}{i(t+1)}}{\log i}$, corresponding to the length of the interval in which Lemma 3 was applied, tends to infinity. Thus it will be necessary to choose N, M such that

$$\log N = o\left(\log \frac{n}{M^2}\right) \tag{9}$$

as $n \rightarrow \infty$.

We now extend both of the initial two sums in the main term of this expression to infinity, so that we can replace it with nC_f .

This introduces another error term of the form $O_A\left(\frac{n}{N}\right)$, to account for all of the new terms $i > N$ in the now infinite sum, and also an error term to account for those additional terms with $i < N$ but where $d > M$. This error term has the form

$$\begin{aligned}
O_A \left(\sum_{i=1}^N \sum_{\substack{d \geq M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \frac{n}{t(t+1)} \prod_{p < i} \frac{p-1}{p} \right) &= O_A \left(\sum_{i=3}^N \frac{n}{i^2 \log i} \sum_{\substack{d \geq M \\ P^+(d) < i}} \frac{1}{d} \right) \\
&= O_A \left(\frac{n}{\log N} \sum_{\substack{d \geq M \\ P^+(d) < N}} \frac{1}{d} \right) \\
&= O_A \left(\frac{n\rho(v)}{\log v} \right).
\end{aligned}$$

Here $v = \frac{\log M}{\log N}$ and we have used Lemma 4 to bound the sum (assuming that N and M satisfy the conditions of that lemma.) Also by Lemma 4 this same expression bounds the size of $\Theta(n, N, M)$, so putting this into our estimate (8) the sum over

$f(k, n)$ becomes

$$nC_f + O_A \left(\sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(\frac{n \log i}{t^2 \exp((1+o(1))u_t \log u_t)} + \frac{n^\epsilon}{t^{2\epsilon}} \right) + \frac{n}{N} + \frac{n\rho(v)}{\log v} \right). \quad (10)$$

We can write the first term inside the innermost sum above as

$$\log i \left(\frac{n}{t^2} \right)^{1-(1+o(1))\frac{\log u_t}{\log i}} < \log N \left(\frac{n}{t^2} \right)^{1-(1+o(1))\frac{\log u_t}{\log N}} < \log N \left(\frac{n}{t^2} \right)^{1-(1+o(1))\frac{\log u'}{\log N}}$$

where $u' = \frac{\log \frac{n}{N^2 M^2}}{\log N}$ is obtained by taking both i and t as large as possible in u_t , $i = N$, $t = NM$. So long as N and M are chosen so that $\frac{\log u'}{\log N} \rightarrow 0$ then this term will dominate the second term in the sum for sufficiently large n (assuming $\epsilon < 1$). We can thus bound the triple sum as

$$\begin{aligned} & \sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} \left(\frac{n \log i}{t^2 \exp((1+o(1))u_t \log u_t)} + \left(\frac{n}{t^2} \right)^\epsilon \right) \\ & \ll n^{1-(1+o(1))\frac{\log u'}{\log N}} \log N \sum_{i=1}^N \sum_{\substack{d < M \\ P^+(d) < i}} \sum_{t \in [id, (i+1)d)} t^{-2+(2+o(1))\frac{\log u'}{\log N}} \\ & \ll n^{1-(1+o(1))\frac{\log u'}{\log N}} \log N \sum_{i=1}^N i^{-2+(2+o(1))\frac{\log u'}{\log N}} \sum_{\substack{d < M \\ P^+(d) < i}} d^{-1+(2+o(1))\frac{\log u'}{\log N}} \\ & \ll n^{1-(1+o(1))\frac{\log u'}{\log N}} \log N \sum_{d < M} d^{-1+(2+o(1))\frac{\log u'}{\log N}} \\ & \ll n^{1-(1+o(1))\frac{\log u'}{\log N}} M^{(2+o(1))\frac{\log u'}{\log N}} \\ & \ll n \exp \left((-1+o(1))(\log n - 2 \log M) \frac{\log u'}{\log N} + \log \log N \right). \quad (11) \end{aligned}$$

Putting this into (10) and approximating $\rho(v) = \exp((-1+o(1))v \log v)$,

$$\begin{aligned} & \sum_{k=1}^n f(k, n) = nC_f \\ & + O_A \left(\frac{n}{\exp \left((1+o(1)) \log \left(\frac{n}{M^2} \right) \frac{\log u'}{\log N} - \log \log N \right)} + \frac{n}{N} + \frac{n}{\exp \left((1+o(1))v \log v \right)} \right). \quad (12) \end{aligned}$$

Finally, we optimize the error term in the expression above by taking $M = n^{1/3-\epsilon}$, and $N = \exp \left(\sqrt{\left(\frac{1}{6} - \epsilon \right) \log n \log \log n} \right)$. Note that this choice satisfies (9) as well as the hypotheses of Lemma 4. This choice gives $\log u' = \left(\frac{1}{2} + o(1) \right) \log \log n$, so,

for sufficiently large n ,

$$\begin{aligned} (1+o(1))(\log n - 2 \log M) \frac{\log u'}{\log N} - \log \log N &= (1+o(1)) \frac{\frac{1}{6} + \epsilon}{\sqrt{\frac{1}{6} - \epsilon}} \sqrt{\log n \log \log n} \\ &> \sqrt{\frac{1}{6} \log n \log \log n} \end{aligned}$$

and

$$(1+o(1))v \log v = \frac{(1+o(1)) \left(\frac{1}{3} - \epsilon\right) \log n}{\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n}} \left(\frac{1}{2} \log \log n\right) > \sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n}.$$

This gives us our result, namely

$$\sum_{k=1}^n f(k, n) = nC_f + O_A \left(n \exp \left(-\sqrt{\left(\frac{1}{6} - \epsilon\right) \log n \log \log n} \right) \right).$$

□

REFERENCES

1. R. Angelo, *A Cameron and Erdős conjecture on counting primitive sets*, *Integers* **18** (2018), Paper No. A25, 4.
2. M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss, *Multiplicative structures in additively large sets*, *J. Combin. Theory Ser. A* **113** (2006), no. 7, 1219–1242.
3. P. J. Cameron and P. Erdős, *On the number of sets of integers with various properties*, *Number theory* (Banff, AB, 1988), de Gruyter, Berlin, 1990, pp. 61–79.
4. A. Chadozeau, *Sur les partitions en chaînes du graphe divisoriel*, *Period. Math. Hungar.* **56** (2008), no. 2, 227–239.
5. P. Erdős and É. Saias, *Sur le graphe divisoriel*, *Acta Arith.* **73** (1995), no. 2, 189–198.
6. J. Friedlander and H. Iwaniec, *Opera de cribro*, American Mathematical Society Colloquium Publications, vol. 57, American Mathematical Society, Providence, RI, 2010.
7. R. Graham, D. Knuth, and O. Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science.
8. H. Liu, P. P. Pach, and R. Palincza, *The number of maximum primitive sets of integers*, arXiv preprint arXiv:1805.06341 (2018).
9. P. Mazet, *Recouvrements hamiltoniens de certains graphes*, *European J. Combin.* **27** (2006), no. 5, 739–749.
10. N. McNew, *On sets of integers which contain no three terms in geometric progression*, *Math. Comp.* **84** (2015), no. 296, 2893–2910.
11. M. B. Nathanson and K. O’Byrant, *On sequences without geometric progressions*, *Integers* **13** (2013), Paper No. A73, 5.
12. C. Pomerance, *On the longest simple path in the divisor graph*, *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing* (Boca Raton, Fla., 1983), vol. 40, 1983, pp. 291–304.
13. R. A. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, *Proc. Roy. Soc. Edinburgh Sect. A* **65** (1960/1961), 332–344 (1960/61).
14. J. Riddell, *Sets of integers containing no n terms in geometric progression*, *Glasgow Math. J.* **10** (1969), 137–146.
15. É. Saias, *Étude du graphe divisoriel. III*, *Rend. Circ. Mat. Palermo (2)* **52** (2003), no. 3, 481–488.
16. G. Tenenbaum, *Integers with a large friable component*, *Acta Arith.* **124** (2006), no. 3, 287–291.
17. ———, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, Providence, RI, 2015, Translated from the 2008 French edition by Patrick D. F. Ion.

18. The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 8.1)*, 2018, <http://www.sagemath.org>.
19. S. Vijay, *On large primitive subsets of $\{1, 2, \dots, 2n\}$* , arXiv preprint arXiv:1804.01740 (2018).

APPENDIX A. RANGES OF VALUES USED IN COMPUTATIONS

i	d up to
5	3600000
6	1000000
7	32000
8	6400
9	2160
10	1176
11	625
12	405
13	270
14	189
15	169
16	160
17	119
18	112
19	88
20	44
$21 \leq i \leq 26$	30
$27 \leq i \leq 30$	25
$31 \leq i \leq 35$	14
$36 \leq i \leq 40$	11
$41 \leq i \leq 50$	10
$51 \leq i \leq 60$	9
$61 \leq i \leq 70$	8
$71 \leq i \leq 80$	7
$81 \leq i \leq 90$	6
$91 \leq i \leq 100$	5

TABLE 1. The values of $w(d, t)$ were computed for all d up to and including the value in the table for each value of i in order to estimate the constant η in the count of maximal primitive sets.

i	d up to
12	6144
$13 \leq i \leq 15$	1536
$16 \leq i \leq 20$	1152
$21 \leq i \leq 24$	256
25	150
$26 \leq i \leq 30$	16
$31 \leq i \leq 40$	12
$41 \leq i \leq 55$	8
$56 \leq i \leq 60$	7
$61 \leq i \leq 75$	4
$76 \leq i \leq 100$	3
$101 \leq i \leq 250$	1

TABLE 2. Values of $g(d, t)$ were computed for all d up to and including the value in the table for each value of i in order to estimate the constant θ in the count of geometric progression free sets.

i	d up to
6	10000
7	120
8	36
9	32
10	14
11	12
12	8
13	8
14	7
15	6
16	6
17	6
18	6

TABLE 3. Values of $v(d, t)$ were computed for all d up to and including the value in the table for each value of i to compute the lower bound for the constant c in the path cover problem.

d	t
1	2
8	$98 \leq t < 105$
10	$96 \leq t < 98$
20	$192 \leq t < 196$
243	$1536 \leq t < 1600$
1215	$7680 \leq t < 8000$
6075	$38400 \leq t < 40000$

TABLE 4. Values of d and t found in the range of d in Table 3 where d was not divisible by 6, but where $v(d, t) = 0$.