Efficient Realization of Nonzero Spectra by Polynomial Matrices

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Nathan McNew

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Advisor: Nicholas Ormes

Author: Nathan McNew

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Abstract

A theorem of Boyle and Handelman gives the necessary and sufficient conditions for an n-tuple of nonzero complex numbers to be the nonzero spectrum of some matrix with nonnegative entries, but is not constructive and puts no bound on the necessary dimension of the matrix. We look instead at polynomial matrices and attempt to reprove the Boyle Handelman theorem in a constructive way, with a bound on the size of the polynomial matrix required to realize a given polynomial.

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Chapter 1

Introduction

In 1949 KR Suleimanova[4] posed the question: Given an $n - tuple$ of complex numbers $\sigma := (\lambda_1, \lambda_2, \cdots \lambda_n)$ when is σ the spectrum of some $n \times n$ matrix A with nonnegative entries? (ie $\det(I - At) = \prod_{i=1}^{n}(t - \lambda_i)$) This problem has come to be known as the Nonnegative Inverse Eigenvalue Problem. When such a matrix A exists we say that A realizes σ. [2]

A matrix is **primitive** if it is a square matrix and some power of it is a matrix with strictly positive entries. The Nonnegative Inverse Eigenvalue Problem is generally studied in terms of primitive matrices, and given the conditions for an $n - tuple$ to be realized by a primitive matrix, they can be easily extended to the general case. There are several known necessary conditions for σ to be realizable by a primitive matrix:

1. $\exists \lambda_i \in \sigma$ such that $\lambda_i \in \mathbb{R}_+$ and $\lambda_i > |\lambda_j| \;\; j \neq i$.

2. $\sigma = \bar{\sigma}$ (For every complex number in σ, its complex conjugate is also in σ.

3. The kth moment of σ, $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$. $\forall k \in \mathbb{N}$ and if $s_k > 0$ then $s_{nk} > 0 \;\; \forall n \in \mathbb{N}$

The first condition comes as a result of the Perron-Froebenius theorem which states that a primitive matrix A must have such an eigenvalue. The second is simply a result of the fact that in order for $\det(ItA)$ to have real coefficients any complex roots must come in conjugate pairs. Finally, the third condition is found from looking at the trace of A^k which is non-negative, and if A^k has a positive trace, then A^{nk} does as well.

As a result of the second condition, the problem can be reformulated as: Given a polynomial $p(t) \in \mathbb{R}[t]$ then is there a matrix A such that $p(t)$ is the characteristic polynomial of A? (ie $p(t) = \det(It - A)) = \prod_{i=1}^{n}(t - \lambda_i)$ In this case σ is the set of the roots of the polynomial, and the roots must still satisfy the first and third conditions.

Boyle and Handelman proved in 1991[1] that as long as the above conditions are fulfilled then there exists an N such that the set σ augmented by N zeros can be realized by a nonnegative primitive matrix. (A matrix is primitive if it is a square nonnegative matrix, and some power of it is strictly positive) The proof is not constructive however, and does not give any bounds on the size of N however, and thus puts no restriction on the size of the matrix.[3]

In terms of polynomials the Boyle Handelman theorem states that given a polynomial $p(t) \in \mathbb{R}[t]$ then there is an integer N and a matrix A such that $t^N p(t)$ is the characteristic polynomial of A. Alternatively one can look at $q(t) = \det(I - tA) = \prod_{i=1}^{n}(1 - \lambda_i t)$ in which case the roots of $q(t)$ are $1/\lambda_i$. In this case, the Boyle Handelman theorem answer's the question: Given a polynomial $q(t) \in t\mathbb{R}[t]$ when does there exist a matrix A such that $\det(I - tA) = q(t)$?

Chapter 2

Graphs and Polynomial Matrices

Let G be a weighted directed graph on N vertices with weights in $\mathbb{R}_+$. Then the **adjacency matrix** M of G is the $N \times N$ matrix in which the (i, j) element is the weight of the edge running from vertex i to vertex j. The **characteristic polynomial** of this matrix (and of the associated graph G) is the polynomial $\chi_M(t) = \det(It - M)$.

The **reverse characteristic polynomial** of the graph is the polynomial $\chi_M^{-1}(t) = \det(I - Mt)$. Note that $\chi_M^{-1}(t) = t^N \chi_M(t^{-1})$ (where M is an $N \times N$ matrix)

As we will show, the directed graph G can also be represented by a polynomial matrix $A[t]$ over $t\mathbb{R}_+[t]$. (Thus the entries of A will consist of polynomials with non-negative coefficients without constant terms besides 0).

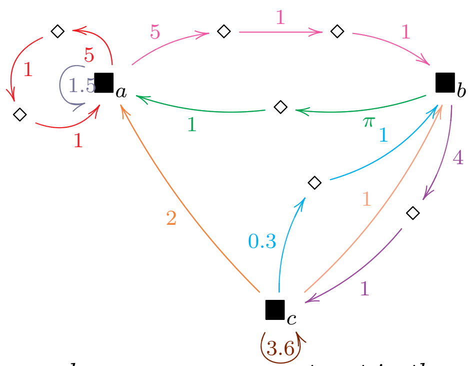
Construction of G: Given an $N \times N$ Polynomial Matrix $A(t)$ over $t\mathbb{R}_+[t]$ the corresponding directed graph G can be constructed as follows: Assign N vertices the labels $1, 2, ...N$. Then for each term $a_n t^p$ in the polynomial in the (i, j) position of $A(t)$, construct a path of length p from vertex i to vertex j, in which the first edge is weighted a_n, and each additional edge (if $p > 1$) is weighted 1. If $p > 1$ then the $p - 1$ additional vertices in the new path are disjoint from the original N

vertices and from vertices used in any other path.

Example 2.0.1. *Take for example the following arbitrarily chosen polynomial matrix over $t\mathbb{R}_+[t]$:*

$$\begin{bmatrix} 5t^3 + 1.5t & 9t^3 & 0 \\ \pi t^2 & 0 & 4t^2 \\ 2t & 0.3t^2 + t & 3.6t \end{bmatrix}$$

From this matrix, by the method described above we can construct the graph below: (Large squares denote the primary vertices, diamonds denote secondary, connecting vertices)



Having constructed this graph, we can now construct in the normal way an adjacency matrix for the graph:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1.5 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 3.6 & 0 & 0.3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi & 0 & 0 & 4 & 0 \end{bmatrix}$$

And compute both the characteristic and reverse characteristic polynomials of this adjacency matrix and graph:

$$\chi_G(t) = t^{10} - 5.1t^9 + 5.4t^8 - 9t^7 + 22.8t^6 - (9\pi + 1.8)t^5 + (32.4\pi - 52)t^4 + 6.0t^3$$

$$\chi_G^{-1}(t) = 6t^7 + (32.4\pi - 52)t^6 - (9\pi + 1.8)t^5 + 22.8t^4 - 9t^3 + 5.4t^2 - 5.1t + 1$$

A matrix can be similarly constructed from a graph, in which chains of vertices with both in and out degree one (meaning there is exactly one edge directed towards the vertex and exactly one edge leaving the vertex) correspond to a term with exponent the length of the chain, and coefficient equal to the product of the weights of paths along the chain.

Theorem 2.0.2. *Given two matricies M over $\mathbb{R}_+$ and $A(t)$ over $t\mathbb{R}_+[t]$ that correspond to the same graph G, then:*

$$\chi_M^{-1}(t) = \det(I - Mt) = \det(I - A[t])$$

Proof. For any i, j such that $A(t)_{i,j}$ is a polynomial of degree greater than 1, then for each term $a_{i,j,n}t^n$, $n > 1$ and $a_{i,j,n} > 0$ there is a path in the graph from vertex i to vertex j of length n, and thus $n + 1$ rows (indexed k_1 ... k_{n+1}) in the matrix M corresponding to each of the $n+1$ vertices along this path. (Note that k_1 corresponds to vertex i in $A(t)$ and k_{n+1} corresponds to vertex j) Each of these rows and columns (except k_1 and k_{n+1}) will have only one nonzero term, in the (k_h, k_{h+1}) position and $a_{i,j,n} = \prod_{h=1}^{n} M_{k_h,k_{h+1}}$.

Each of these additional $n-1$ rows can be removed from the matrix $I - Mt$ without changing the determinant by the following row operations, working backwards from $h = n$ to 2:

1. Subtract row k_h scaled by the entry in position (k_{h-1}, k_h) from row $k_h - 1$.

2. Subtract from column k_{n+1}Column k_h is multiplied by the entry in position (k_h, k_{n+1}).

This sequence results in the product of the terms in positions (k_{h-1}, k_h) and (k_h, k_{n+1}) appearing in position (k_{h-1}, k_{n+1}) and only a 1 remaining in both row and column k_h. Thus after repeating this process for all the intermediate nodes, there will be a term equivalent to the product of their weights times t raised to the length of the chain added to the (k_1, k_{n+1}) position, and a 1 in the primary diagonal for each row/column associated with each intermediate node. The determinant can be expanded by minors at each of these 1s, thus reducing the size of the matrix.

Repeating this process for each such $a_{i,j,n}$ term in $I - A(t)$ (and switching rows as necessary at the end) will produce the matrix $I - A(t)$ from $I - Mt$ without changing the determinant.

$\square$

Chapter 3

Our Approach

Our approach is to study the Nonnegative Inverse Eigenvalue Problem and specifically the Boyle Handelman Theorem in terms of polynomial matrices rather than matrices over $\mathbb{R}_+$. Since there is currently no known bound on the number of zeros that must be appended to the set of complex numbers to be realized by the Boyle Handelman theorem, there is no known "efficient construction" (In terms of size) of a matrix over $\mathbb{R}_+$ that realizes a polynomial which satisfies the necessary conditions.

We attempt, rather, to reprove the Boyle Handelman conjecture by constructing an "efficient" polynomial matrix (in terms of the size of the matrix, without any bound on the degree of polynomials used in that matrix) that realizes a given polynomial. If we were able to bound both the size of the matrix and the degree of the polynomials used then we would be able to bound the size of the corresponding matrix over $\mathbb{R}_+$. Here we focus only on trying to bound the size of the polynomial matrix and make no attempt to control the degrees of the polynomials. In this vein we will make use of polynomials which are truncations of the power series for $p(t)^{1/N}$.

By strengthening the third of the necessary conditions so that $\forall k \in \mathbb{N}, s_k > 0$ we have the following important result:

Theorem 3.0.3. *Assume that $p(t) = \prod_{i=1}^{d}(1-\lambda_i t)$ where the $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ satisfy the necessary conditions and the strengthened third condition above. Then there is an $N \geq 1$ such that the power series expansion for $p(t)^{1/N}$ is of the form*

$$p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k$$

where $r_k \geq 0$ for all $k \geq 1$.

Proof. Write:

$$p(t) = \prod_{i=1}^{d}(1-\lambda_i t)$$

Recall that the power series expansion for $(1-t)^{1/N}$ is given by:

$$(1-t)^{1/N} = \sum_{k=0}^{\infty} \binom{1/N}{k} t^k$$

Where:

$$\binom{1/N}{k} = \frac{1/N(1/N-1)(1/N-2)\cdots(1/N-k+1)}{k(k-1)(k-2)\cdots 1}$$

Then:

$$\begin{aligned}
p(t)^{1/N} &= \prod_{i=1}^{d}(1-\lambda_i t)^{1/N} \\
&= \prod_{i=1}^{d}\left(\sum_{k=0}^{\infty}\binom{1/N}{k}(-\lambda_i)^k t^k\right) \\
&= \prod_{i=1}^{d}\left(1 - \sum_{k=1}^{\infty}\left|\binom{1/N}{k}\right| \lambda_i^k t^k\right)
\end{aligned}$$

The kth coeffcient of this looks like

$$r_k = \left|\binom{1/N}{k}\right| \left(\lambda_1^k + \lambda_2^k + \cdots + \lambda_d^k\right) + \sum (-1)^l \left|\binom{1/N}{k_1}\binom{1/N}{k_2}\cdots\binom{1/N}{k_d}\right| \lambda_{i_1}^{k_1}\lambda_{i_2}^{k_2}\cdots\lambda_d^{k_d}$$

Where the second sum ranges over combinations of $k_i \geq 0$ such that $k_1 + k_2 + \cdots +$

$k_d = k$, $l \geq 2$ is the count of nonzero k_i, and $k_{i_1}, k_{i_2} \cdots k_{i_l}$ are these nonzero values.

Factoring out $\left|\binom{1/N}{k}\right| \lambda_1^k$ (and assuming that λ_1 is the perron eigenvalue) The first term becomes

$$1 + \left(\frac{\lambda_2}{\lambda_1}\right)^k + \cdots + \left(\frac{\lambda_d}{\lambda_1}\right)^k$$

which approaches 1 as $k \to \infty$ and is always positive (by BH+). Therefore, this term has a uniform lower bound $\delta > 0$ (which does not depend on k or N).

The absolute value of the second term is at most

$$\sum \left| \frac{\binom{1/N}{k_1}\binom{1/N}{k_2}\cdots\binom{1/N}{k_d}}{\binom{1/N}{k}} \right| \left|\frac{\lambda_1}{\lambda_1}\right|^{k_1} \left|\frac{\lambda_2}{\lambda_1}\right|^{k_2} \cdots \left|\frac{\lambda_d}{\lambda_1}\right|^{k_d}$$

Now for $l \geq 2$ and $N \geq 2$:

$$\begin{aligned}
\left| \frac{\binom{\frac{1}{N}}{k_1}\binom{\frac{1}{N}}{k_2}\cdots\binom{\frac{1}{N}}{k_d}}{\binom{\frac{1}{N}}{k}} \right| &= \left| \frac{\left(\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_1+1)}{k_1!}\right)\left(\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_2+1)}{k_2!}\right)\cdots\left(\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k_d+1)}{k_d!}\right)}{\left(\frac{\frac{1}{N}(\frac{1}{N}-1)\cdots(\frac{1}{N}-k+1)}{k!}\right)} \right| \\
&= \left| \left(\frac{1}{N}\right)^{l-1} \left(\frac{k!}{k_1!k_2!\cdots k_d!}\right) \frac{\left((\frac{1}{N}-1)\cdots(\frac{1}{N}-k_1+1)\right)\cdots\left((\frac{1}{N}-1)\cdots(\frac{1}{N}-k_d+1)\right)}{\left((\frac{1}{N}-1)\cdots(\frac{1}{N}-k+1\right)} \right| \\
&< \left| \left(\frac{1}{N}\right)^{l-1} \left(\frac{k!}{k_1!k_2!\cdots k_d!}\right) \frac{(k_{i_1}-1)!(k_{i_2}-1)!\cdots(k_{i_l}-1)!}{(k-1)!} \right| \\
&= \left| \left(\frac{1}{N}\right)^{l-1} \left(\frac{k}{k_{i_1}k_{i_2}\cdots k_{i_l}}\right) \right| \\
&= \left| \left(\frac{1}{N}\right) \left(\frac{k}{k_{i_1}k_{i_2}\cdots k_{i_l}N^{l-2}}\right) \right| \\
&\quad (k_{i_1}k_{i_2}\cdots k_{i_l}N^{l-2}) \text{ is minimized when } l = 2 \text{ and } k_{i_1} = k-1 \\
&< \left| \left(\frac{1}{N}\right) \left(\frac{k}{k-1}\right) \right| \\
&< \frac{2}{N}
\end{aligned}$$

and that

$$\sum \left|\frac{\lambda_1}{\lambda_1}\right|^{k_1} \left|\frac{\lambda_2}{\lambda_1}\right|^{k_2} \cdots \left|\frac{\lambda_d}{\lambda_1}\right|^{k_d} < \left(\frac{1}{1-\left|\frac{\lambda_2}{\lambda_1}\right|}\right)\left(\frac{1}{1-\left|\frac{\lambda_3}{\lambda_1}\right|}\right)\cdots\left(\frac{1}{1-\left|\frac{\lambda_d}{\lambda_1}\right|}\right) = M$$

by expanding the right hand side into a product of geometric series. Therefore, there is a uniform upper bound of the form $\frac{2}{N}M$ where M does not depend on k or N.

Then all we need to do is choose N such that $\delta > \frac{2}{N}M$ $\square$

Using this result we make the following conjecture:

Conjecture 1. *Let $p(t)$ be a polynomial which satisfies the condition that $\exists N \geq 1$ such that $p(t)^{1/N} = 1 - \sum_{k=1}^{\infty} r_k t^k$ where $r_k \geq 0$ for all $k \geq 1$. Then there exists an $N \times N$ polynomial matrix $A(t)$ with all nonnegative coefficients such that* $\det(I - A(t))$ *= p(t).*

As a result of Theorems 1 and 2, proving this conjecture would be (nearly) equivalent to proving the Boyle Handelman theorem (with the exception of the strengthening of the third condition in Theorem 2.) Unlike the Boyle Handelman result, the proof that we are working on would be constructive, and would have a bound on the size of the polynomial matrix required to realize a given polynomial. Without putting a bound on the degree of the polynomial matrix, however, this conjecture does not establish any bounds on the size of the regular matrix over $\mathbb{R}_+$. If, however, the size of the polynomial matrix and the degrees of polynomials used in the matrix could both be bounded, then a bound on the size of the realizing regular matrix could be achieved.

At the moment we are able to prove the above conjecture for the cases $N = 1, 2, 3$.

Chapter 4

Cases N = 1, 2

Case 1 (Proof (N=1))**.** *Trivial. If* $p(t)^1 = 1 - r(t)$ *where* $r(t)$ *has no negative coefficients then the matrix* $A(t) = [r(t)]$ *suffices.*

$$\det(I - A(t)) = \det([1 - r(t)]) = 1 - r(t) = p(t)$$

Case 2 (Proof (N=2))**.** *Suppose* $p(t)^{1/2} = 1 - r(t)$ *where* $r(t)$ *has no negative coefficients. Then let* $q(t)$ *be the polynomial that results when* $r(t)$ *is truncated to degree* n *(*n *greater than or equal to the degree of* $p(t)$*.) Consider the polynomial* $(1 - q(t))^2$*.*

The first n *coefficients of this polynomial will match* $p(t)$*. Let* $R(t) = (1 - q(t))^2 - p(t)$*. Then* $R(t)$ *will be a polynomial with lowest order term of degree n+1 and highest degree of 2n, and is described by:*

$$R(t) = \sum_{i=n+1}^{2n} \sum_{j+k=i} q_j q_k t^i$$

Where q_i *is the coefficient of of the* t^i *term in* $q(t)$*. Since all* q_i *are nonnegative,* $R(t)$ *will contain only nonnegative terms.*

Then construct the matrix:

$$A(t) = \begin{bmatrix} q(t) & \frac{R(t)}{t} \\ t & q(t) \end{bmatrix}$$

$$\det(I - A(t)) = (1 - q(t))^2 - R(t) = p(t)$$

Example 4.0.4 (N = 2)**.** *Consider the polynomial* $p(t) = 1 - 3t - 2t^2 + 4t^3$

The power series of $p(t)^{1/2}$ *is:* $p(t)^{1/2} = 1 - \frac{3t}{2} - \frac{17t^2}{8} - \frac{19t^3}{16} - \frac{517t^4}{128} - \frac{2197t^5}{256} + \cdots$

Let $q(t) = \frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}$

Then $(1 - q(t))^2 = 1 - 3t - 2t^2 + 4t^3 + \frac{517t^4}{64} + \frac{323t^5}{64} + \frac{361t^6}{256}$

and $R(t) = (1 - q(t))^2 - p(t) = \frac{517t^4}{64} + \frac{323t^5}{64} + \frac{361t^6}{256}$

We can then construct the matrix $A(t)$ *as described above:*

$$A(t) = \begin{bmatrix} (\frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}) & (\frac{517t^3}{64} + \frac{323t^4}{64} + \frac{361t^5}{256}) \\ (t) & (\frac{3t}{2} + \frac{17t^2}{8} + \frac{19t^3}{16}) \end{bmatrix}$$

and $A(t)$ *realizes the original polynomial* $p(t) = 1 - 3t - 2t^2 + 4t^3$.

Chapter 5

The case N=3

The $N = 3$ case extends the ideas used in the $N = 2$ case, but is more complex since the "left over" terms of the $(1 - q(t))^3$ term cannot be assumed to be all positive. In this case we work with the matrix:

$$A(t) = \begin{bmatrix} q(t) & \alpha(t) & \beta(t) \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix}$$

Where again $q(t)$ is assumed to be some truncation of the power series $r(t) = 1 - p(t)^{1/3}$ of degree n at least as great as the degree of $p(t)$, although in this case a degree higher than that of $p(t)$ may be necessary. In this case:

$$\det(I - A(t)) = (1 - q(t))^3 - t^2\alpha(t) - t\beta(t)(1 - q(t))$$

Were $R(t) = p(t) - (1 - q(t))^{1/3}$ strictly positive then this remainder could be accommodated by the $\alpha(t)$ term as in the $N = 2$ case, and the $\beta(t)$ term would not be needed, however this is never the case. Consider the highest order term of $R(t)$. This term (of degree $3n$) will be have coefficient $(-q_n)^3$ where q_n is the coefficient multiplying the t^n term in $q(t)$. Thus $R(t)$ will necessarily contain at least 1 negative coefficient, and in practice it usually has many more.

On the other hand it is easy to show that the lowest order term of $R(t)$ will always be positive: Since this term is of order $n+1$, greater than the order of $p(t)$, the coefficient of the term of order $n+1$ in the polynomial $(1-r(t))^3 = p(t)$ must be 0. The only term "missing" of order $n+1$ when expanding $(1-q(t))$ is $3(-r_{n+1})$. Since this term is negative, the corresponding term in $R(t)$ must be positive.

Since negative terms exist in $R(t)$, the $\beta(t)$ polynomial term must be used. Any term $b_m t^m$ in $\beta(t)$ is multiplied by $t(1-q(t))$ in the determinant of $A(t)$ and thus has the effect of decreasing the $(m+1)$th coefficient of $(1-q(t))^3 - t\beta(t)(1-q(t))$ and increasing the $(m+2)$th through $(m+n+1)$th coefficients. The end goal when constructing this $\beta(t)$ polynomial is to result in a remainder polynomial $A(t) = (1-q(t))^3 - t\beta(t)(1-q(t)) - p(t)$ which has all positive coefficients.

To this end, we can take the lowest order term of $R(t)$, which we know to be positive, and include it in $\beta(t)$. This is in a sense the largest that this coefficient of $\beta(t)$ can be, any larger and a negative term would result in $A(t)$, but it also provides the maximum benefit in terms of increasing the coefficients of terms with higher powers. If the next lowest order term of the resulting $A(t)$ is also positive then we can repeat the process, including this term in $\beta(t)$ as well. This process can be continued either until a negative coefficient is reached, or until the entire remaining $A(t)$ is positive. (Success) In the case that a negative coefficient is reached, one can try again with a larger n, ie using a $q(t)$ with a greater number of terms from the power series of $p(t)^{1/3}$.

As an example:

Example 5.0.5 (N=3). *Let* $p(t) = 1 - 5t + 7t^2 - 3t^3$

$$p(t)^{1/2} = 1 - \frac{5t}{2} + \frac{3t^2}{8} - \frac{9t^3}{16} - \frac{189t^4}{128} - \frac{891t^5}{256} \cdots$$

We cannot use a 2×2 matrix since the power series of $p(t)^{1/2}$ is not of the correct form. The power series of $p(t)^{1/3}$ is of the correct form, however:

$$p(t)^{1/3} = 1 - \frac{5t}{3} - \frac{4t^2}{9} - \frac{76t^3}{81} - \frac{508t^4}{243} - \frac{3548t^5}{729} \cdots$$

We let $q(t)$ be this power series truncated to 3 terms:

$$q(t) = \frac{5t}{3} + \frac{4t^2}{9} + \frac{76t^3}{81}$$

$$(1-q(t))^3 = 1-5t+7t^2-3t^3+\frac{508t^4}{81}-\frac{1532t^5}{243}-\frac{3536t^6}{2187}-\frac{32528t^7}{6561}-\frac{23104t^8}{19683}-\frac{438976t^9}{531441}$$

Only the first term of $R(t)$ is positive:

$$R(t) = +\frac{508t^4}{81} - \frac{1532t^5}{243} - \frac{3536t^6}{2187} - \frac{32528t^7}{6561} - \frac{23104t^8}{19683} - \frac{438976t^9}{531441}$$

Including this term as the first term in $\beta(t)$

$$(1-q(t))^3-\frac{508t^4}{81}(1-q(t)) = 1-5t+7t^2-3t^3+\frac{112t^5}{27}+\frac{2560t^6}{2187}+\frac{6080t^7}{6561}-\frac{23104t^8}{19683}-\frac{438976t^9}{531441}$$

Thus we now have an additional positive term which can be included in $\beta(t)$. Repeating this process twice more we eventually get:

$$(1-q(t))^3-\left(\frac{508t^4}{81}+\frac{112t^5}{27}+\frac{17680t^6}{2187}\right)(1-q(t)) = 1-5t+7t^2-3t^3+\frac{106576t^7}{6561}+\frac{41408t^8}{6561}+\frac{3592064t^9}{531441}$$

which is $p(t)$ plus a polynomial with only positive coefficients, which can then be chosen to be $\alpha(t)$ in the matrix. Bringing all of these polynomials together we can *construct the matrix:*

$$A(t) = \begin{bmatrix} \frac{5t}{3} + \frac{4t^2}{9} + \frac{76t^3}{81} & \frac{106576t^5}{6561} + \frac{41408t^6}{6561} + \frac{3592064t^7}{531441} & \frac{508t^3}{81} + \frac{112t^4}{27} + \frac{17680t^5}{2187} \\ 0 & \frac{5t}{3} + \frac{4t^2}{9} + \frac{76t^3}{81} & t \\ t & 0 & \frac{5t}{3} + \frac{4t^2}{9} + \frac{76t^3}{81} \end{bmatrix}$$

such that $A(t)$ realizes the original polynomial $p(t)$.

At this point in our research a computer program was written which ran through the steps of this "greedy algorithm" to determine whether such a matrix could be constructed for trial polynomials, $p(t)$ which satisfied the condition that the power series of $p(t)^{1/3} - 1$ had all negative coefficients. All cubic polynomials with integer coefficients less than 100 were tested and no counter examples were found.

The goal of this algorithm can be reformulated instead as constructing a polynomial $b(t)$,

$$b(t) = \sum_{i=M+1}^{3n} b_i t^i$$

such that $p(t) - (1-q(t))^3 + b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less. Then if $b(t)$ has only positive terms, the realizing matrix can be easily constructed. In the following propositions we demonstrate that it is always possible to construct such a $b(t)$ with all positive coefficients.

First, we note the following:

Proposition 5.0.6. *Let $p(t)^{1/3} = 1 - r(t) = 1 - q(t) - s(t)$, where $q(t)$ is polynomial of degree n, equal to the power series $r(t)$ truncated to degree n, and $s(t)$ is a power series consisting of the remaining terms in $r(t)$. Then*

$$b_m = 3\left[s(t)(1 - q(t) - s(t))\right]_m$$

Proof. By the construction of $b(t)$, $\forall m < 3n$

$$\left[p(t) - (1-q(t))^3 + b(t)(1-q(t))\right]_m = 0$$

$$[b(t)(1-q(t))]_m = \left[(1-q(t))^3 - p(t)\right]_m$$

$$p(t) = [1-q(t)-s(t)]^3 = (1-q(t))^3 - 3s(t)(1-q(t))^2 + 3s(t)^2(1-q(t)) - s(t)^3$$

Plugging this expression in for $p(t)$ above:

$$[b(t)(1-q(t))]_m = \left[3s(t)(1-q(t))^2 - 3s(t)^2(1-q(t)) + s(t)^3\right]_m$$

The lowest order term of $s(t)^3$ will have degree $3n+3$, so this term can be dropped:

$$\begin{aligned}[b(t)(1-q(t))]_m &= \left[3s(t)(1-q(t))^2 - 3s(t)^2(1-q(t))\right]_m \\ &= [(1-q(t))3(s(t))((1-q(t)) - s(t))]_m \\ &= [(1-q(t))\,[3(s(t))((1-q(t)-s(t))]]_m\end{aligned}$$

Thus

$$[b(t)]_m = b_m = 3\,[(s(t))((1-q(t)-s(t))]_m$$

□

Alternatively, we can write this result in terms of $r(t)$ as:

$$b_m = 3\,[s(t)(1-q(t)-s(t))]_m = 3\left[r_m + \sum_{i=1}^{m-n} r_i r_{m-i}\right]$$

Proposition 5.0.7. *Assume $p(t)$ satisfies the Boyle Handelman conditions, as well as our strengthened third condition, that λ_1 is the perron root of $p(t)$ and $p(t)^{1/3} =$*

$1 - r(t)$. *A good estimate of the coefficients* r_n *of* $r(t)$ *is*

$$\left|\binom{1/3}{n}\right| \lambda_1^n (a(1/\lambda_1))^{1/3}$$

where $a(t)$ *is the polynomial*

$$a(t) = \frac{p(t)}{1 - \lambda_1 t}$$

and λ_1 *is the Perron root of* $p(t)$. *By a "good estimate" we mean that*

$$\lim_{n\to\infty} \frac{r_n}{\left|\binom{1/3}{n}\right| \lambda_1^n (a(1/\lambda_1))^{1/3}} = 1$$

Proof.

Subclaim 1. *Let* $\epsilon > 0$ *be given. Then there exists an* $N > 0$ *such that for any* $n > N$ *and for any* $0 < j < n$,

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-j}(1+\epsilon)^j$$

Proof. First note that

$$\begin{aligned}
\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| &= \left|\frac{\left(\frac{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-(n-j-1))}{(n-j)!}\right)}{\left(\frac{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-(n-1))}{n!}\right)}\right| \\
&= \left|\frac{n!}{(n-j)!}\frac{1}{((\frac{1}{3}-(n-j))(\frac{1}{3}-(n-j+1))\cdots(\frac{1}{3}-(n-1)))}\right| \\
&= \frac{n!}{(n-j)!}\frac{1}{\left|\frac{1-3(n-j)}{3}\right|\left|\frac{1-3(n-j+1)}{3}\right|\cdots\left|\frac{1-3(n-1)}{3}\right|} \\
&= \frac{n!}{(n-j)!}\frac{1}{(n-j)\left|1-\frac{1}{3(n-j)}\right|(n-j+1)\left|1-\frac{1}{3(n-j+1)}\right|\cdots(n-1)\left|\frac{1}{3(n-1)}\right|} \\
&= \frac{n}{n-j}\prod_{k=1}^{j}\frac{1}{1-\frac{1}{3(n-k)}}.
\end{aligned}$$

and

$$\log \sqrt[j]{\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}}} = \frac{1}{j}\sum_{k=1}^{j} \log\left(\frac{1}{1-\frac{1}{3(n-k)}}\right)$$

Since the terms $\frac{1}{1-\frac{1}{3(n-k)}}$ increase with k,

$$\begin{aligned} 0 &< \frac{1}{j}\sum_{k=1}^{j} \log\left(\frac{1}{1-\frac{1}{3(n-k)}}\right) \\ &\le \frac{1}{n-1}\sum_{k=1}^{n-1} \log\left(\frac{1}{1-\frac{1}{3(n-k)}}\right) \\ &= \frac{1}{n-1}\sum_{k=1}^{n-1} \log\left(\frac{1}{1-\frac{1}{3k}}\right) \end{aligned}$$

The last expression above is the average of the first $n-1$ terms of the form $\log\left(\frac{1}{1-\frac{1}{3k}}\right)$. Since these terms tend to 0 as $k \to \infty$, the average of them does as well. Thus there exists an N such that $\forall n \ge N$

$$\frac{1}{n-1}\sum_{k=1}^{n-1} \log\left(\frac{1}{1-\frac{1}{3k}}\right) < \log(1+\epsilon)$$

and for $n \ge N$ and for any $1 < j < n$,

$$\log \sqrt[j]{\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}}} < \log(1+\epsilon)$$

Therefore

$$\prod_{k=1}^{j} \frac{1}{1-\frac{1}{3(n-k)}} < (1+\epsilon)^j$$

and

$$\left|\frac{\binom{1/3}{n-j}}{\binom{1/3}{n}}\right| < \frac{n}{n-j}(1+\epsilon)^j$$

□

Subclaim 2. *Let $\epsilon > 0$ be given and fix $K > 0$. There exists an $N > K$ such that* *for any $n > N$ and for any $0 < j < K$,*

$$1 < \left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < 1 + \epsilon$$

Proof. let $\epsilon_1 = (1+\epsilon)^{\frac{1}{k+1}} - 1$

Then by Subclaim 1 there exists an $N_1 > K$ such that $\forall n \geq N_1$ and for every $0 < j < K < N_1$

$$\left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < \frac{n}{n-j}(1+\epsilon_1)^j \leq \frac{n}{n-K}(1+\epsilon_1)^K$$

Since $\lim_{n\to\infty} \frac{n}{n-K} = 1$ There exists N_2 such that $\forall n \geq N_2$

$$\frac{n}{n-K} \leq (1 - \epsilon_1)$$

Let $N = \max(N_1, N_2)$. Then $\forall n \geq N$ and $0 < j < K$

$$\left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < \frac{n}{n-K}(1+\epsilon_1)^K < (1+\epsilon_1)(1+\epsilon_1)^K = (1+\epsilon_1)^{K+1} = (1+\epsilon)$$

□

We use these two subclaims to show that given $\epsilon > 0$ there exists an N such that $\forall n > N$

$$\left| \frac{r_n}{\left|\binom{1/3}{n}\right| \lambda_1^n a(1/\lambda_1)^{1/3}} - 1 \right| < \epsilon$$

Let $\alpha(t) = 1 + \sum_{i=1}^{\infty} \alpha_i t^i$ denote the power series expansion for $a(t)^{1/3} = (p(t)/(1-\lambda_1 t))^{1/3}$ at $t = 0$. Then

$$p(t)^{1/3} = 1 - r(t) = (1-\lambda_1 t)^{1/3}\alpha(t) = \left(1 - \sum_{i=1}^{\infty} \left|\binom{1/3}{i}\right| \lambda_1^i t^i\right)\left(1 + \sum_{i=1}^{\infty} \alpha_i t^i\right)$$

We can then write r_n as

$$r_n = \left|\binom{1/3}{n}\right| \lambda_1^n - \alpha_n + \sum_{k=1}^{n-1} \left|\binom{1/3}{n-k}\right| \alpha_k \lambda_1^{n-k}$$
$$= \left|\binom{1/3}{n}\right| \lambda_1^n \left(1 + \sum_{k=1}^{n-1} \left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| \alpha_k \lambda_1^{-k} - \frac{\alpha_n}{\left|\binom{1/N}{n}\right|} \lambda_1^{-n}\right)$$

Let $\delta = \frac{\epsilon a(1/\lambda_1)^{1/3}}{5}$.

If λ_2 is the root of $a(t) = p(t)/(1-\lambda_1 t)$ with the greatest modulus (ie $\forall \lambda_i$ roots of $a(t)$, $|\lambda_2| \geq |\lambda_i|$) then the power series $\alpha(t) = a(t)^{1/3}$ has radius of convergence $1/|\lambda_2|$ which is greater than $1/\lambda_1$. Now for some $K > 0$ and $n > K_1$ we can write

$$1 + \sum_{k=1}^{n-1} \left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| \alpha_k \lambda_1^{-k} - \frac{s_n}{\left|\binom{1/3}{n}\right|} \lambda_1^{-n} = a(1/\lambda_1)^{1/3} \tag{5.0.1}$$
$$+ \left(1 + \sum_{k=1}^{K} \alpha_k \lambda_1^{-k} - a(1/\lambda_1)^{1/3}\right) \tag{5.0.2}$$
$$+ \sum_{k=1}^{K} \left(\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| - 1\right) \alpha_k \lambda_1^{-k} \tag{5.0.3}$$
$$+ \sum_{k=K+1}^{n-1} \left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| \alpha_k \lambda_1^{-k} \tag{5.0.4}$$
$$- \frac{\alpha_n}{\left|\binom{1/3}{n}\right|} \lambda_1^{-n} \tag{5.0.5}$$

We can now make each of the terms (5.0.2) through (5.0.5) small as follows:

(5.0.2): Since $1/\lambda_1$ lies in the radius of convergence of $\alpha(t)$, $1 + \sum_{k=1}^{K_1} \alpha_k \lambda_1^{-k}$ converges to $a(1/\lambda_1)^{1/3}$ So for some $K_1 > 0$,

$$\left|1 + \sum_{k=1}^{K_1} \alpha_k \lambda_1^{-k} - a(1/\lambda_1)^{1/3}\right| < \delta$$

(5.0.4) is less than

$$\sum_{k=K+1}^{n-1} \left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| |\alpha_k| \lambda_1^{-k}$$

Fix ϵ_2 such that $\frac{1+\epsilon_2}{\lambda_1} < \frac{1}{|\lambda_2|}$

Then by Subclaim 1 there exists a K_2 such that $\forall n > K_2$ and $j < n$

$$\left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < \frac{n}{n-j}(1+\epsilon)^j$$

Then $\forall n > K_2$

$$\begin{aligned}
\sum_{k=K_2+1}^{n-1} \left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| |\alpha_k| \lambda_1^{-k} &< \sum_{k=K_2+1}^{n-1} \frac{n}{n-k}(1+\epsilon_2)^k |\alpha_k| \lambda_1^{-k} \\
&= \sum_{k=K_2+1}^{n-1} \left(1 - \frac{k}{n-k}\right) |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k \\
&< \sum_{k=K_2+1}^{n-1} |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k + \sum_{k=K_2+1}^{n-1} k |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k
\end{aligned}$$

$\frac{1+\epsilon_2}{\lambda_1}$ lies within the radius of convergence of both of these series, since $\alpha(t)$ converges absolutely thus there exists a $K_3 \geq K_2$ such that $\forall n > k_3$ both

$$\sum_{k=K_3+1}^{n-1} |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k < \delta$$

$$\sum_{k=K_3+1}^{n-1} k |\alpha_k| \left(\frac{1+\epsilon_2}{\lambda_1}\right)^k < \delta$$

Thus

$$\sum_{k=K_3+1}^{n-1} \left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| |\alpha_k| \lambda_1^{-k} < 2\delta$$

(5.0.5)For sufficiently large n

$$
\begin{aligned}
\frac{\alpha_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n} &\leq \left|\left(\frac{n!}{\frac{1}{3}(\frac{1}{3}-1)\cdots(\frac{1}{3}-(n-1))}\right)\right| |\alpha_n| \lambda_1^{-n} \\
&= \left|\frac{1}{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}\right| \left|\frac{2}{\frac{1}{3}-3}\right| \cdots \left|\frac{n-2}{\frac{1}{3}-(n-1)}\right| (n-1)(n) |\alpha_n| \lambda_1^{-n} \\
&< \frac{27}{10}(n-1)(n) |\alpha_n| \lambda_1^{-n}
\end{aligned}
$$

The series $\sum(n-1)(n)\alpha_n t^n$ has radius of convergence greater than $1/\lambda_1$, and converges absolutely, so the sequence $(n-1)(n)\alpha_n t^n$ is cauchy. Thus there exists K_5 such that $\forall n > K_5$

$$
\frac{\alpha_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n} < \frac{27}{10}(n-1)(n) |\alpha_n| \lambda_1^{-n} < \delta
$$

At this point we fix K in the equation above so that $K = \max(K_1, K_2, K_3, K_4, K_5)$, and look at the remaining term:

(5.0.3) is less than

$$
\sum_{k=1}^{K_1} \left(\left| \frac{\binom{1/3}{n-k}}{\binom{1/3}{n}} \right| - 1 \right) |s_k| \lambda_1^{-k}
$$

Let

$$
\epsilon_2 = \frac{\delta}{\sum_{k=1}^{K_1} |\alpha_k| \lambda_1^{-k}}
$$

Then by Subclaim 2, since K is fixed, there exists an $N > K$ such that $\forall n > N$ and $0 < j \leq K$

$$
1 < \left| \frac{\binom{1/3}{n-j}}{\binom{1/3}{n}} \right| < 1 + \epsilon_2
$$

Thus, $\forall n > N$

$$\sum_{k=1}^{K_1}\left(\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right| - 1\right)|\alpha_k|\,\lambda_1^{-k} < \sum_{k=1}^{K_1}((1+\epsilon_2)-1)\,|\alpha_k|\,\lambda_1^{-k} = \epsilon_2\sum_{k=1}^{K_1}|\alpha_k|\,\lambda_1^{-k} = \delta$$

Combining the above, for $K = \max(K_1, K_2, K_3, K_4, K_5)$ and $n > N$

$$1 + \sum_{k=1}^{n-1}\left|\frac{\binom{1/3}{n-k}}{\binom{1/3}{n}}\right|\alpha_k\lambda_1^{-k} - \frac{\alpha_n}{\left|\binom{1/3}{n}\right|}\lambda_1^{-n} < a(1/\lambda_1)^{1/3} + 5\delta = a(1/\lambda_1)^{1/3}(1+\epsilon)$$

So

$$\left|\frac{r_n}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}} - 1\right| = \left|\frac{\left|\binom{1/3}{n}\right|\lambda_1^n\left(1+\sum_{k=1}^{n-1}\left|\frac{\binom{1/N}{n-k}}{\binom{1/N}{n}}\right|\alpha_k\lambda_1^{-k} - \frac{\alpha_n}{\left|\binom{1/N}{n}\right|}\lambda_1^{-n}\right)}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}} - 1\right|$$

$$< \left|\frac{\left|\binom{1/3}{n}\right|\lambda_1^n\left(a(t)^{1/3}(1+\epsilon)\right)}{\left|\binom{1/3}{n}\right|\lambda_1^n a(1/\lambda_1)^{1/3}} - 1\right| = \epsilon$$

□

Proposition 5.0.8. *Let $1 - c(t) = (p(t))^{2/3}$. Then there exists an N such that for $k > N$, $c_k \geq 0$.*

Proof. By the same method as above, a good approximation for c_n is

$$\left|\binom{2/3}{n}\right|\lambda_1^n(q(1/\lambda_1))^{2/3}$$

.

Note that $q(1/\lambda_1)$ must be positive since $q(0) = 1$ and $q(t)$ has no root between 0 and $1/\lambda_1$). □

We can now return to our polynomial $b(t)$, constructed such that $p(t) - (1 -$

$q(t))^3 + b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less. (n is the degree of $q(t)$)

From Proposition 5.0.6,

$$[b(t)]_m = b_m = 3\left[(s(t))((1-q(t)-s(t))\right]_m$$

where $p(t)^{1/3} = 1 - r(t) = 1 - q(t) - s(t)$. We can write

$$\begin{aligned} p(t)^{2/3} = 1 - c(t) &= (1-q(t)-s(t))^2 \\ &= 1 - 2q(t) - 2s(t) + 2q(t)s(t) + q(t)^2 + s(t)^2 \end{aligned}$$

Thus for $n < m \le 2n$

$$b_m = 3\left[(s(t))((1-q(t)-s(t))\right]_m \;= \frac{3}{2}[c_n + \left(q(t)^2\right)_n]$$

and for $2n < m \le 3n$

$$b_m = 3\left[(s(t))((1-q(t)-s(t))\right]_m \;= \frac{3}{2}[c_n - \left(s(t)^2\right)_n]$$

So if n is large enough so that $c_m \ge 0$ for $m \ge n$, we have

$$b_m = 3\left[s(t)(1-q(t)-s(t))\right]_m = \frac{3}{2}[c_m + \left(q(t)^2\right)_m] \ge 0$$

So it remains to show that

$$b_m = 3\left[s(t)(1-q(t)-s(t))\right]_m = \frac{3}{2}[c_m - \left(s(t)^2\right)_m] \ge 0$$

for $2n < m \le 3n$.

From Propositions 2 and 3 above we can use the approximations $s_n \approx \left|\binom{1/3}{n}\right| \lambda_1^n (q(1/\lambda_1))^{1/3}$

and $c_n \approx \left|\binom{2/3}{n}\right| \lambda_1^n (q(1/\lambda_1))^{2/3}$.

Note that For $2n < m \leq 3n$

$$\sum_{i,m-i>n} \left|\binom{1/3}{i}\binom{1/3}{m-i}\right| \leq \sum_{i,m-i>n} \left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|$$
$$= (m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|$$

Proposition 5.0.9. *For $2n < m \leq 3n$ there exists a $1 > d > 0$ such that*

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{\left|\binom{2/3}{m}\right|} \leq 1-d$$

Proof.

Subclaim 3. *For a fixed value of n this expression,*

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{\left|\binom{2/3}{m}\right|}$$

Is strictly increasing in the range $2n < m < 3n$

Proof. The denominator of this term, $\left|\binom{2/3}{m}\right|$ is strictly decreasing for increasing n. We can show that the denominator of this term is strictly increasing by looking at the ratio of 1 term to the next:

$$\frac{(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|}{(m-2n)\left|\binom{1/3}{n+1}\binom{1/3}{m-n)}\right|} = \left|\frac{(m-2n-1)(m-n)}{(m-2n)(1/3-(m-n-1))}\right|$$
$$= \left|(1-\frac{1}{m-2n})(\frac{m-n}{4/3-(m-n)})\right|$$
$$= \frac{1-\frac{1}{m-2n}}{1-\frac{4/3}{m-n}}$$

Then we can compare $\frac{1}{m-2n}$ to $\frac{4/3}{m-n}$, by looking at their ratio:

$$\frac{\frac{4/3}{m-n}}{\frac{1}{m-2n}} = \frac{4(m-2n)}{3(m-n)}$$

This term is strictly increasing in the range $2n < m < 3n$ and is equal to 0 when $m = 2n$ and $4/6$ when $m = 3n$, thus for $2n < m < 3n$ we have $\frac{1}{m-2n} > \frac{4/3}{m-n}$ and $1 - \frac{1}{m-2n} < 1 - \frac{4/3}{m-n}$ so the ratio $\frac{1-\frac{1}{m-2n}}{1-\frac{4/3}{m-n}}$ is less than 1, demonstrating that for $2n < m < 3n$

$$(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right| < (m-2n)\left|\binom{1/3}{n+1}\binom{1/3}{m-n)}\right|$$

$\square$

Thus it suffices to consider the largest possible value of m, $3n$ which gives us

$$\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1)}\right|}{\left|\binom{2/3}{3n}\right|}$$

Subclaim 4. *For all $n \geq 1$*

$$\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1)}\right|}{\left|\binom{2/3}{3n}\right|} < \frac{(n)\left|\binom{1/3}{n}\binom{1/3}{2n)}\right|}{\left|\binom{2/3}{3n}\right|}$$

Proof. Again by looking at their ratio, the denominators cancel leaving:

$$\begin{aligned}
\frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1)}\right|}{(n)\left|\binom{1/3}{n}\binom{1/3}{2n)}\right|} &= \frac{(n-1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1)}\right|}{(n)\left|\binom{1/3}{n}\binom{1/3}{2n)}\right|} \\
&= \frac{n-1}{n}\frac{1/3-n}{n+1}\frac{2n}{1/3-2n-1} \\
&= \frac{n-1}{n}\frac{n(1-\frac{1}{3n})}{n+1}\frac{2n}{(2n-1)(1-\frac{1}{3(2n-1)})} \\
&= \frac{(n-1)(2n)}{(n+1)(2n-1)}\frac{(1-\frac{1}{3n})}{(1-\frac{1}{3(2n-1)})}
\end{aligned}$$

Now we can observe that

$$\frac{1-\frac{1}{3n}}{1-\frac{1}{3(2n-1)}} < 1$$

$$\frac{(n-1)(2n)}{(n+1)(2n-1)} = \frac{2n^2-2n}{2n^2+n-1} < 1 \quad (n \geq 1)$$

and thus their product is less than 1. □

Subclaim 5. *The terms*

$$\frac{(n)\left|\binom{1/3}{n}\binom{1/3}{2n)}\right|}{\left|\binom{2/3}{3n}\right|}$$

are strictly decreasing for increasing values of n.

Proof. We use the same trick of comparing the ratio of one term of this series to the one following it and find:

$$\frac{\frac{(n+1)\left|\binom{1/3}{n+1}\binom{1/3}{2n-1)}\right|}{\left|\binom{2/3}{3n}\right|}}{\frac{(n)\left|\binom{1/3}{n}\binom{1/3}{2n-1)}\right|}{\left|\binom{2/3}{3n}\right|}} = \frac{n+1}{n}\frac{1/3-n}{n+1}\frac{((1/3)-2n)(1/3-(2n+1))}{(2n+1)(2n+2)}\times$$

$$\frac{(3n+1)(3n+2)(3n+3)}{((2/3)-3n)((2/3)-(3n+1))((2/3)-(3n+2))}$$

$$= \frac{(1-\frac{1}{3n})\frac{2n}{2n+2}((1-\frac{1}{6n})(1-\frac{1}{6n+3})}{\frac{3n}{3n+3}(1-\frac{2}{9n})(1-\frac{2}{9n+3})(1-\frac{2}{9n+6})}$$

We now define

$$f(x) = \frac{(1-\frac{1}{3x})((1-\frac{1}{6x})(1-\frac{1}{6x+3})}{(1-\frac{2}{9x})(1-\frac{2}{9x+3})(1-\frac{2}{9x+6})} \quad = \frac{3(3x-1)(6x-1(3x+1)(3x+1)(3+2)}{x(2x+1)(9x-1)(9x-2)(9x+4)}$$

We can find the derivitive of this function:

$$f'(x) = \frac{d}{dx}\left(\frac{3(3x-1)(6x-1(3x+1)(3x+1)(3+2)}{x(2x+1)(9x-1)(9x-2)(9x+4)}\right)$$

$$= \frac{6(104976x^7+130491x^6+49167x^5-1485x^4-4239x^3-258x^2+140x+8)}{x^2(1458x^4+1215x^3+135x^2-70x-8)^2}$$

The numerator of this function factors as $6(3x+1)(8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6)$ Thus the only roots of $f'(x)$ can be $x=-\frac{1}{3}$ or where

$$(8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6)=0$$

This has no solutions in $[1,\infty)$ since

$$(8+116x-606x^2-2421x^3+5778x^4+31833x^5+34992x^6)$$
$$= (34992x^6-2421x^3)+(31833x^5-606x^2)++5778x^4+116x+8$$

and each term above is strictly positive for $n \geq 1$. By a similar argument, the denominator of $f'(x)$ has no roots in $[1, \infty)$. We can calculate $f'(1) = \frac{1672800}{7452900} \approx 0.2244 > 0$ thus $f'(x) > 0$ for all $x \in [1, \infty)$. Thus f(x) is strictly increasing on $[1, \infty)$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{(1 - \frac{1}{3x})((1 - \frac{1}{6x})(1 - \frac{1}{6x+3})}{(1 - \frac{2}{9x})(1 - \frac{2}{9x+3})(1 - \frac{2}{9x+6})} = 1$$

so $f(x) < 1$ for all $x \in [1, \infty)$. The terms

$$\frac{(n) \left| \binom{1/3}{n} \binom{1/3}{2n)} \right|}{\left| \binom{2/3}{3n} \right|}$$

are strictly decreasing for increasing values of n. $\square$

We can then evaluate this expression for $n = 1$ and find

$$\frac{\left| \binom{1/3}{1} \binom{1/3}{2)} \right|}{\left| \binom{2/3}{3} \right|} = \frac{3}{4}$$

Thus for all n and $2n < m \leq 3n$

$$\frac{(m - 2n - 1) \left| \binom{1/3}{n+1} \binom{1/3}{m-(n+1))} \right|}{\left| \binom{2/3}{m} \right|} \leq \frac{3}{4} = 1 - \frac{1}{4}$$

so we can choose $d = \frac{1}{4}$ and the proposition is valid. $\square$

Then write

$$c_m - \left(s(t)^2\right)_m = c_m - \sum_{i, m-i > n} s_i s_{m-i}$$

For convenience we define $A = \left| \binom{2/3}{n} \right|$ and $B = (m - 2n - 1) \left| \binom{1/3}{n+1} \binom{1/3}{m-(n+1))} \right|$. Choose $\delta > 0$ such that $Ad > \delta(A - 2B - \delta B)$

By the propositions above, we can choose n such that $\forall m > n$, $c_m > (1 -$

$\delta)\left|\binom{2/3}{m}\right|\lambda_1^m(q(1/\lambda_1))^{2/3}$ and $s_m < (1+\delta)\left|\binom{1/3}{m}\right|\lambda_1^m(q(1/\lambda_1))^{1/3}$. So

$$
\begin{aligned}
c_m - \left(s(t)^2\right)_m > {} & (1-\delta)\left|\binom{2/3}{n}\right|\lambda_1^n(q(1/\lambda_1))^{2/3} \\
& - \sum_{i,m-i>n}\left((1+\delta)\left|\binom{1/3}{i}\right|\lambda_1^i(q(1/\lambda_1))^{1/3}\right)\left((1+\delta)\left|\binom{1/3}{m-i}\right|\lambda_1^{m-i}(q(1/\lambda_1))^{1/3}\right) \\
= {} & (1-\delta)\left|\binom{2/3}{n}\right|\lambda_1^n(q(1/\lambda_1))^{2/3} - \sum_{i,m-i>n}(1+\delta)^2\left|\binom{1/3}{i}\right|\left|\binom{1/3}{m-i}\right|\lambda_1^m(q(1/\lambda_1))^{2/3} \\
= {} & \left(\lambda_1^n(q(1/\lambda_1))^{2/3}\right)\left((1-\delta)\left|\binom{2/3}{n}\right| - (1+\delta)^2\sum_{i,m-i>n}\left|\binom{1/3}{i}\right|\left|\binom{1/3}{m-i}\right|\right) \\
\geq {} & \left(\lambda_1^n(q(1/\lambda_1))^{2/3}\right)\left((1-\delta)\left|\binom{2/3}{n}\right| - (1+\delta)^2(m-2n-1)\left|\binom{1/3}{n+1}\binom{1/3}{m-(n+1)}\right|\right)
\end{aligned}
$$

In terms of A and B defined above, the term in parentheses can be expanded to:

$$A - \delta A - B - 2\delta B - \delta^2 B$$

Then since $\frac{B}{A} \leq 1-d$, $A - B \geq Ad$ and the expression above is greater than or equal to

$$Ad - \delta A - 2\delta B - \delta^2 B = Bd - \delta(A - 2B - \delta B)$$

By our choice of δ above this is strictly greater than or equal to 0, so we are done.

Namely, this demonstrates that we can construct a polynomial $b(t)$ of degree at most $3n$ with positive coefficients such that $p(t) - (1-q(t))^3 + b(t)(1-q(t))$ has coefficient 0 for all terms with degree $3n$ or less.

Let $d(t) = (1-q(t))^3 - b(t)(1-q(t)) - p(t)$ Since n is the degree of $q(t)$, $q(t)^3$ will have degree $3n$ as well, so any remaining terms in $d(t)$ will be the result of trailing terms in the product of $b(t)$ and $q(t)$. Since both of these polynomials contain only positive coefficients, $d(t)$ will as well. As a result, $p(t) = (1-q(t))^3 - b(t)(1-q(t)) -$

$d(t)$ and we can construct the matrix

$$A(t) = \begin{bmatrix} q(t) & \frac{d(t)}{t^2} & \frac{b(t)}{t} \\ 0 & q(t) & t \\ t & 0 & q(t) \end{bmatrix}$$

such that $I - A(t)$ has determinant $p(t)$.

Chapter 6

Further Work

The obvious next step in this research would be to continue to study this problem for larger values of N, and to develop constructions for correspondingly larger polynomial matrices. Already for the case $N = 4$ at least a slightly new method will be required. The logical progression to a 4×4 matrix would be to construct

$$A(t) = \begin{bmatrix} q(t) & \alpha(t) & \beta(t) & \gamma(t) \\ 0 & q(t) & t & 0 \\ 0 & 0 & q(t) & t \\ t & 0 & 0 & q(t) \end{bmatrix}$$

In this case $I - A(t)$ has determinant

$$(1 - q(t))^4 - \alpha(t)t^3 - \beta(t)(1 - q(t))t^2 - \gamma(t)(1 - q(t))^2 t$$

Ignoring the $\gamma(t)$ term, (ie letting $\gamma(t) = 0$) results in a problem identical to the N=3 case, however it does not appear that this method will suffice for all polynomials which satisfy the condition that $p(t)^{1/4}$ has all negative coefficients. Thus it is likely that a solution will require use of the $\gamma(t)$ polynomial, however the same "greedy" algorithm cannot be used. Whereas $(1 - q(t))$ had all negative coefficients, except for the leading 1, meaning that each coefficient of $\beta(t)$ "helped" all of the coefficients of higher order by making them more positive, $(1 - q(t))^2$ will not in general have that property, so coefficients of $\lambda(t)$ would correct some terms while "hindering" others by making them more negative. It is also possible that a different matrix configuration, utilizing more of the positions occupied by ts and 0s is required.

Clearly, the ideal result would be a general proof that demonstrated this result for all N.

Another interesting possibility for research would be to look at the degrees of polynomials required in this construction and to attempt to constrain them. As mentioned before, if a given construction could control the size of both the polynomial matrix and the degrees of the polynomials used in the matrix, then it would put a constraint on the required size of the "normal" matrix over $\mathbb{R}_+$ described in the original problem.

Interestingly, the results given here for $N = 1$ and 2 already constrain the degree of the polynomials used. (For a polynomial of degree d, the $N = 1$ requires only a polynomial of degree d and the $N = 2$ case requires a polynomial of degree at most $2d$) However, the polynomials required in the $N = 3$ case may currently have arbitrarily high degree.

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