Avoiding Geometric Progressions in the Integers

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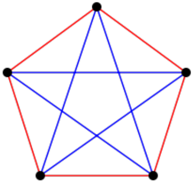
Extremal Number Theory

Extremal combinatorics studies how large a collection of objects can be before some property or structure must exist.

For example: In a complete graph on n vertices with red and blue edges, what is the largest n can be if there are no monochromatic triangles? **5**



Extremal number theory asks similar questions about the integers.

Arithmetic Progressions

How many consecutive integers can be colored red and blue if no three equally spaced terms have the same color? **8**

1 2 3 4 5 6 7 8

An **arithmetic progression** (**AP**) of length k is a set:

$$\{a, a+b, a+2b, \cdots, a+(k-1)b\}$$

Theorem (Van der Waerden, 1927)

Color $\mathbb{N}$ using a finite palette of colors. No matter how this is done there will always be an arbitrarily long, monochromatic, arithmetic progression.

Roth's Theorem

In 1936, Erdős and Turán conjectured that any subset of $\mathbb{N}$ with a positive proportion of the integers has arbitrarily long arithmetic progressions.

$$\text{Density} \qquad d(A) = \lim_{N\to\infty} \frac{|A \cap [1, N]|}{N}$$

$$\text{Upper Density} \qquad \bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1, N]|}{N}$$

Theorem (Roth, 1953)

If $A \subset \mathbb{N}$ has $\bar{d}(A) > 0$ then A contains arithmetic progressions of length 3.

Later generalized by Szemerédi (1975) to progressions of arbitrary length, proving Erdős and Turán's conjecture.

How large of a set can we construct while avoiding 3-term APs?

Sets free of 3-term-arithmetic progressions

Greedy set, A_3^*. Include n in A_3^* if doing so does not create a 3-term-AP involving terms already included in A_3^*.

$$\begin{aligned} A_3^* &= \{0, 1, 3, 4, 9, 10, 12, 13, 27 \cdots\} \\ &= \{n \geq 0 \mid n \text{ has no digit 2 in its base 3 representation}\} \end{aligned}$$

$$|A_3^* \cap [1, N]| \approx N^{\log_3 2}$$

One can do much better. It is possible to construct sets up to N free of 3-term-APs of size:

$$\frac{1}{\log^{1/4} N} \cdot \frac{N}{2^{2\sqrt{2\log_2 N}}} \text{ (Behrend, 1946)}$$

$$\frac{N \log^{1/4} N}{2^{2\sqrt{2\log_2 N}}} \text{ (Elkin, 2008)}$$

Upper bound of sets free of arithmetic progressions

These constructions remain far short of the known upper bounds.

As $N \to \infty$ how large can a subset of $[1, N]$ be before we know that it must contain a 3 AP?

- $\frac{N}{\log\log N}$ (Roth, 1954)
- $\frac{N}{\log^c N}$ for some constant $c > 0$ (Heath-Brown, 1987)
- $\frac{N}{\log^{1/20} N}$ (Szemeredi, 1990)
- $\frac{N(\log\log N)^{1/2}}{\log^{1/2} N}$ (Bourgain, 1999)
- $\frac{N(\log\log n)^2}{\log^{2/3} N}$ (Bourgain, 2008)
- $\frac{N(\log\log N)^5}{\log N}$ (Sanders, 2011)
- $\frac{N(\log\log N)^4}{\log N}$ (Bloom, 2014)

$5000 Problem

Conjecture (Erdős)

Any set A for which

$$\sum_{n \in A} \frac{1}{n} = \infty$$

contains arbitrarily long arithmetic progressions.

Sets free of geometric progressions

In 1961, Rankin suggested looking at sets free of geometric progressions.

A 3-term-geometric progression (**GP**) is a set of integers $\{a, ar, ar^2\}$, $r \in \mathbb{Q}$. For example, $\{1, 2, 4\}$ $\{2, 6, 18\}$ or $\{4, 6, 9\}$.

The set of square free numbers, S, is free of geometric progressions, and $d(S) = \frac{6}{\pi^2} \approx 0.6079$. Roth's theorem is false for geometric progressions.

Sets free of geometric progressions

Write $v_p(n)$ to denote the number of times that the prime p divides n.

For example, $v_2(40) = 3$, $v_3(40) = 0$, $v_5(40) = 1$.

If $\{a, b, c\}$ is a geometric progression, then for every prime, p, $\{v_p(a), v_p(b), v_p(c)\}$ forms an arithmetic progression.

Using this, Rankin constructs the set

$$G_3^* = \{n \in \mathbb{N} : \text{for all primes } p, v_p(n) \in A_3^*\}$$

which is free of geometric progressions. (A_3^* is the set free of arithmetic progressions obtained by the greedy algorithm.)

Rankin's Set

$$\begin{aligned} G_3^* &= \{n \in \mathbb{N} : \text{for all primes } p, v_p(n) \in A_3^*\} \\ &= \{1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19 \cdots\} \end{aligned}$$

Brown and Gordon showed that Rankin's set is the set obtained by greedily including integers without creating a geometric progression. Its density is

$$d(G_3^*) = \prod_p \left(\frac{p-1}{p} \sum_{i \in A_3^*} \frac{1}{p^i} \right) = \frac{1}{\zeta(2)} \prod_{i>0} \frac{\zeta(3^i)}{\zeta(2 \cdot 3^i)} = 0.71974\ldots$$

What is the greatest possible density of a geometric progression free set?

Notation

Define:

$$\overline{\alpha} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is GP-free}\}$$
$$\alpha = \sup\{d(A) : A \subset \mathbb{N} \text{ is GP-free and } d(A) \text{ exists}\}$$

Rankin: $0.71974 \leq \alpha$

$$\alpha \leq \overline{\alpha} \leq \frac{7}{8}$$

(a odd $\Rightarrow$ must exclude one of a, $2a$, $4a$.)

Upper Bounds

The upper bound for the upper density of a GP-free set has been improved several times.

- $\overline{\alpha} \leq \frac{6}{7} \approx 0.8571$ (Riddell, 1969; Beiglböck, Bergelson, Hindman and Strauss, 2006)
- $\overline{\alpha} < 0.8688$ (Brown and Gordon, 1996)
- $\overline{\alpha} < 0.8495$ (Nathanson and O'Bryant, 2013)
- $\overline{\alpha} < 0.8339$ (Claimed by Riddell, 1969 but stated "The details are too lengthy to be included here.")

Theorem (M., 2013)

The constant $\overline{\alpha}$ is effectively computable, and satisfies

$$0.730027 < \overline{\alpha} < 0.772059.$$

Avoiding s-smooth progressions

Say that a geometric progression $\{a, ar, ar^2\}$ is s-smooth if the common ratio $r \in \mathbb{Q}$, involves only primes at most s.

Then we define

$$\overline{\alpha_s} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is free of } s\text{-smooth rational GPs}\}.$$

$$\overline{\alpha} \leq \overline{\alpha_s} \text{ for any } s \geq 2.$$

The argument earlier shows that $\overline{\alpha_2} \leq 7/8$.

Theorem (Nathanson and O'Bryant, 2013)

$\overline{\alpha_2} = 0.846378\ldots$ *(and is irrational.)*

Bounding $\overline{\alpha_3}$

Idea: the first seven 3-smooth numbers, $\{1, 2, 3, 4, 6, 8, 9\}$, contain the 4 progressions $(1, 2, 4), (2, 4, 8), (1, 3, 9)$ and $(4, 6, 9)$ which cannot all be precluded by removing any single number.

Thus for each $(b, 6) = 1$ at least two of $\{b, 2b, 3b, 4b, 6b, 8b, 9b\}$ must be excluded. $\Rightarrow \overline{\alpha} \leq \overline{\alpha_3} \leq \frac{25}{27}$.

In general: Compute the largest subset of the 3-smooth integers up to k free of geometric progressions. If it requires an additional number to be excluded to avoid 3-smooth GPs, we get a better upper bound for $\overline{\alpha_3}$.

Bounding $\overline{\alpha_3}$

k	# of exclusions		k	# of exclusions		k	# of exclusions
4	1		243	13		1458	25
9	2		256	14		1728	26
16	3		288	15		1944	27
18	4		384	16		2048	28
32	5		486	17		2304	29
36	6		512	18		2592	30
64	7		576	19		3072	31
81	8		729	20		3888	32
96	9		864	21		4096	33
128	10		972	22		4374	34
144	11		1024	23		5184	35
192	12		1296	24		5832	36

$$\overline{\alpha_3} < 1 - \frac{1}{3}\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{18}+\frac{1}{32}+\cdots+\frac{1}{5832}\right) \approx 0.791266$$

Computing $\overline{\alpha}$

Idea: Stitch together bounds for $\overline{\alpha_s}$ with Rankin's construction for primes greater than s.

$$\overline{\alpha_s} \prod_{p>s} \left(\frac{p-1}{p} \sum_{i \in A_3^*} p^{-i} \right) \leq \overline{\alpha} \leq \overline{\alpha_s}$$

So, $\lim_{s\to\infty} \overline{\alpha_s} = \overline{\alpha}$.

Theorem (M., 2013)

For each ϵ with $0 < \epsilon < 1$, the constant $\overline{\alpha}$ can be computed to within ϵ in time

$$O\left(1.6538^{(-2\log_2 \epsilon)^{\frac{1}{\epsilon}}}\right)$$

Other Geometric Progressions

Integer Ratios: $\overline{\beta} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is integer ratio GP-free}\}$

- $0.75 \le \overline{\beta}$ (Beiglböck, Bergelson, Hindman and Strauss)
- $0.815509 < \overline{\beta} < 0.819222$ (M.)
- $0.818410 < \overline{\beta}$ (Ford)

Real Numbers: Nathanson and O'Bryant construct an integer ratio GP-free subset of $[0, 1]$ with measure greater than 0.815509.

$\mathbb{Z}/n\mathbb{Z}$ **(monoid under multiplcation)**:
Largest GP-free subset has size $\ll \frac{n(\log\log n)^4}{(\log n)^{\frac{1}{2}}}$ (M.)

Gaussian Integers, Other Number Fields:
Williams REU 2014 (To appear)

The end

Thank you